CS231 Algorithms | **Handout # 5**
Prof. Lyn Turbak | **September 12, 2001**
Wellesley College

Asymptotics

Motivation

- Fine-grained bean counting exposes too much detail for comparing functions.
- Want a course-grained way to compare functions that ignores constant factors and focuses on their relative growth in the limit as input sizes get large.
- For example, consider:

	$n = 1$	$n = 1,000$	$n = 1,000,000$
$p(n) = 100n + 1000$			
$q(n) = 3n^2 + 2n + 1$			
$r(n) = 0.1n^2$			

Sketch the above functions on the same set of axes:

How Do Your Functions Grow?

Asymptotic notation is a way of characterizing functions that facilitates comparing their growth in the limit of large inputs. Here is an informal summary of the notation:

Notation	Pronunciation	Loosely
$f \in \omega(g)$	f is way bigger than g	$f > g$
$f \in \Omega(g)$	f is at least as big as g	$f \geq g$
$f \in \Theta(g)$	f is about the same as g	$f = g$
$f \in O(g)$	f is at most as big as g	$f \leq g$
$f \in o(g)$	f is way smaller than g	$f < g$

Notes:

- Each of $\omega(g)$, $\Omega(g)$, $\Theta(g)$, $O(g)$, $o(g)$ denotes a *set* of functions. Thus, $\omega(g)$ is the set of all functions way bigger than g, $\Omega(g)$ is the set of all functions at least as big as g, etc.
- The notation $f = \omega(g)$ is really shorthand for $f \in \omega(g)$, and similarly for Ω, Θ, O, o.
- The phrases "is at least $O(\ldots)$" and "is at most $\Omega(\ldots)$" are non-sensical. "Is at least" should be written Ω and "is at most" should be written O.

Intuitively, what are the relationships between p, q, and r?

Relating the Notations

Here are some of the relationships between the notations:

1. If $f \in \omega(g)$, then $f \in \Omega(g)$.
2. If $f \in o(g)$, then $f \in O(g)$.
3. $\Omega(g) \supset (\omega(g) \cup \Theta(g))$.
4. $O(g) \supset (o(g) \cup \Theta(g))$.
5. $\Theta(g) = (\Omega(g) \cap O(g)$.
6. $f \in \omega(g)$ iff $g \in o(f)$.
7. $f \in \Omega(g)$ iff $g \in O(f)$.
8. $f \in \Theta(g)$ iff $g \in \Theta(f)$.

Warning: unlike numbers, not every pair of functions is comparable! (see the later example).

Use a Venn diagram to depict relationships 1–5.

Formalizing o and ω

$f \in o(g)$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 0$
$f \in \omega(g)$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \infty$

Examples:

- Show $p \in o(r)$.
- Show $r \in \omega(p)$.

Formalizing O

$$\boxed{\begin{array}{lll} O(g) & = & \text{The set of all functions } f \text{ such that} \\ & & \quad \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\ & & \qquad 0 \leq f(n) \leq c \cdot g(n) \text{ for all } n \geq n_0. \end{array}}$$

This can also be expressed more succinctly in purely mathematical notation as:

$$\boxed{O(g) \;=\; \{f \mid \exists\, c > 0, n_0 > 0 \;.\; \forall\, n \geq n_0 \;.\; 0 \leq f(n) \leq c \cdot g(n)\}}$$

Think of this as a game. If you claim that $f \in O(g)$, then you must select a particular c and n_0. Then your oppenent tries to find a particular n that defeats your claim.

Examples of O

- Show $p \in O(q)$.
  1. use $c = 1$, $n_0 = 1000$.
  2. use $c = 1000$, $n_0 = 1$.
- Can we show $q \in O(p)$?
- Show $r \in O(q)$ (use $c = 1$ $n_0 = 1$).
- Show $q \in O(r)$ (use $c = 40$; what must n_0 be?).

Formalizing Ω and Θ

$$\boxed{\begin{array}{lll} \Omega(g) & = & \text{The set of all functions } f \text{ such that} \\ & & \quad \text{there exist positive constants } c \text{ and } n_0 \text{ such that} \\ & & \qquad 0 \leq c \cdot g(n) \leq f(n) \text{ for all } n \geq n_0. \end{array}}$$

$$\boxed{\begin{array}{lll} \Theta(g) & = & \text{The set of all functions } f \text{ such that} \\ & & \quad \text{there exist positive constants } c_1,\, c_2 \text{ and } n_0 \text{ such that} \\ & & \qquad 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \text{ for all } n \geq n_0. \end{array}}$$

A Trick for Showing Θ

The following fact is often handy for showing that two functions are related by Θ:

$$\boxed{\text{If } \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = k > 0, \text{ then } f \in \Theta(g).}$$

Example: Use the above fact to show that $q \in \Theta(r)$ and $r \in \Theta(q)$.

The converse of the above limit trick is not true. That is, although the limit trick works most of the time to show that two functions are related by Θ, there are some Θ relationships that cannot be shown by the limit trick. E.g., $f(n) = 2 + sin(n)$ and $g(n) = 2$.

Does anything Fall Between the Cracks?

The Venn diagram relating ω, Ω, Θ, O, and o implies that there are functions that are $O(g)$ that are neither $o(g)$ nor $\Theta(g)$, and there are functions that are $\Omega(g)$ that are neither $\omega(g)$ nor $\Theta(g)$.

Here's a concrete example:

- $f(n) = \frac{1}{n}$
- $g(n) = n$
- $h(n) = n^{\sin(n)}$

Show that $h \in O(g)$, but $h \notin o(g)$ and $h \notin \Theta(g)$. (Similarly, $h \in \Omega(f)$, but $h \notin \omega(f)$ and $h \notin \Theta(f)$.

Incomparable Functions

Not every two functions are comparable via ω, Ω, Θ, O, and o.

Example: Show that $k(n) = \sqrt{n}$ is unrelated to $h(n)$ above.

Exponentials

Notation:

- $a^n =$ the product of n copies of a.
- $a^{-n} = \frac{1}{a^n}$.

Key Identities:

- $a^m \cdot a^n = a^{m+n}$. (Special case: $a^0 = 1$.)
- $(a^m)^n = a^{m \cdot n} = (a^n)^m$.

Examples:

- $(5^2)^3 =$
- $5^2 \cdot 5^3 =$
- $5^2 + 5^3 =$
- $25^{\frac{3}{2}} =$

Relating Exponentials

Suppose:

$$\begin{aligned} f(n) &= 2^n \\ g(n) &= 3^n \\ h(n) &= 2^{cn} \\ k(n) &= 2^{c+n} \end{aligned}$$

What symbols can fill the following blanks?

1. $g \in \quad\quad (f)$
2. $h \in \quad\quad (f)$ $(c < 1)$
3. $h \in \quad\quad (f)$ $(c = 1)$
4. $h \in \quad\quad (f)$ $(c > 1)$
5. $k \in \quad\quad (f)$ (any$c$)

Logarithms

Notation:

- $\log_b(a)$ = the power to which b must be raised to equal a. (More loosely, it is the number of times that a can be divided by b to reach 1.)
- $\log_b(1/a) = -log_b(a)$
- $\lg n = \log_2 n$
- $\ln n = \log_e n$
- $\log_b^k(n) = (\log_b n)^k$

Key Identities (duals of exponential identities):

- $\log_c(a \cdot b) = \log_c(a) + \log_c(b)$
  - Special case: $\log_c(1) = 0$.
  - Special case: $\log_c(a^n) = n \cdot \log_c(a)$
- $\log_c(a) = \log_c(b) \cdot \log_b(a)$

Examples:

- $\lg(2n^3) =$
- $\ln(32) =$

Relating Exponentials and Logarithms

Key Identity:

- $b^{(\log_b(a))} = a = log_b(b^a)$

Examples:

- $\lg \sqrt[3]{4} =$
- $32^{(\lg(n))} =$

Asymptotics Involving Exponentials and Logarithms

- How do $\log_2 n$ and $\log_3 n$ compare?
- How do 2^n and 3^n compare?
- Fact 1: if $a > 1$, $\lim_{n\to\infty} \left(\frac{a^n}{n^b}\right) = \infty$. (Can show this via l'Hôpital's rule.)

  Fact 1 implies $a^n \in \omega(n^b)$ if $a > 1$. In other words: *Any exponential with base* > 1 *grows faster than any polynomial.*
- Substituting $\lg n$ for n and 2^a for a in Fact 1 yields:

  Fact 2: if $a > 0$, $\lim_{n\to\infty} \left(\frac{n^a}{\lg^b n}\right) = \infty$.

  Fact 2 implies $n^a \in \omega(\lg^b n)$ if $a > 0$. In other words: *Any positive polynomial grows faster than any polylogarithmic function.*

Factorials

- Definition: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$
- Stirling's approximation: $n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$
- Asymptotics derivable from Stirling's approximation:
  - $n! = o(n^n)$
  - $n! = \omega(2^n)$
  - $\lg(n!) = \Theta(n \cdot \lg n)$