Geometric transformations

Translation by the vector $(d_x, d_y, d_z)^\top$:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{pmatrix}$$

Translation matrix: $$T(d_x, d_y, d_z) = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometric transformations

Scaling by the factors s_x, s_y, s_z:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \\ s_z \cdot z \\ 1 \end{pmatrix}$$

Scaling matrix: $S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Geometric transformations

Rotation around the z-axis by the angle θ:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation matrix: $$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometric transformations

Rotation around the x-axis by the angle θ:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation matrix: $$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometric transformations

Rotation around the y-axis by the angle θ:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation matrix: $$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometric transformations

Rotation around an arbitrary axis by the angle θ:

- Shift the rotation by a translation such that it passed through the origin.
- Rotation around the z-axis, such that the rotation axis is mapped to the y/z-plane.
- Rotation around the x-axis, such that the rotation axis is mapped to the z-axis.
- Rotation by the angle θ around the z-axis.
- Reverse the three first transformations.

$$T(-d_x, -d_y, -d_z) \circ R_z(-\theta_z) \circ R_x(-\theta_x) \circ R_z(\theta) \circ R_x(\theta_x) \circ R_z(\theta_z) \circ T(d_x, d_y, d_z)$$

Geometric transformations

As already in the case of 2D graphics, the composition of transformations can be implemented by matrix multiplication.

The last line for all above mentioned matrices is $(0, 0, 0, 1)$. Matrix multiplication preserves this property.

In the two-dimensional case there is exactly one transformation matrix that maps three noncollinear points to three other noncollinear points.

In the three-dimensional case there exists exactly one transformation matrix that maps four noncoplanar points to four other noncoplanar points.

Given four noncoplanar points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \in \mathbb{R}^3$ and the target points $\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3, \mathbf{p}'_4$, the transformation matrix is obtained by solving the system of linear equations

$$\mathbf{p}'_i \;=\; M \cdot \mathbf{p}_i \qquad (i = 1, 2, 3, 4)$$

(in homogeneous coordinates) where

Geometric transformations

$$M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this sense, transformations can be interpreted as changing from one coordinate system to another.