Homogeneous arithmetic

- Legal operations always end in 0 or 1!

$$\text{vector+vector:}\quad \begin{bmatrix}\vdots\\0\end{bmatrix}+\begin{bmatrix}\vdots\\0\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\0\end{bmatrix}$$

$$\text{vector-vector:}\quad \begin{bmatrix}\vdots\\0\end{bmatrix}-\begin{bmatrix}\vdots\\0\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\0\end{bmatrix}$$

$$\text{scalar*vector:}\quad s\begin{bmatrix}\vdots\\0\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\0\end{bmatrix}$$

$$\text{point+vector:}\quad \begin{bmatrix}\vdots\\1\end{bmatrix}+\begin{bmatrix}\vdots\\0\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\1\end{bmatrix}$$

$$\text{point-point:}\quad \begin{bmatrix}\vdots\\1\end{bmatrix}-\begin{bmatrix}\vdots\\1\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\0\end{bmatrix}$$

$$\text{point+point:}\quad \begin{bmatrix}\vdots\\1\end{bmatrix}+\begin{bmatrix}\vdots\\1\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\2\end{bmatrix}$$

$$\text{scalar*point:}\quad s\begin{bmatrix}\vdots\\1\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\s\end{bmatrix}$$

$$\left\{\begin{matrix}\text{weighted average}\\\text{affine combination}\end{matrix}\right\}\text{ of points:}\quad \frac{1}{3}\begin{bmatrix}\vdots\\1\end{bmatrix}+\frac{2}{3}\begin{bmatrix}\vdots\\1\end{bmatrix}\Rightarrow\begin{bmatrix}\vdots\\1\end{bmatrix}$$

Homogeneous point transform

- Transform a point:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx}p_x + m_{xy}p_y + m_{xz}p_z + d_x \\ m_{yx}p_x + m_{yy}p_y + m_{yz}p_z + d_y \\ m_{zx}p_x + m_{zy}p_y + m_{zz}p_z + d_z \\ 0+0+0+1 \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \vec{\mathbf{d}}$$

  - Top three rows are the affine transform!
  - Bottom row stays 1

Homogeneous vector transform

- Transform a vector:

$$\begin{bmatrix} v'_x \\ v'_y \\ v'_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} m_{xx}v_x + m_{xy}v_y + m_{xz}v_z + 0 \\ m_{yx}v_x + m_{yy}v_y + m_{yz}v_z + 0 \\ m_{zx}v_x + m_{zy}v_y + m_{zz}v_z + 0 \\ 0+0+0+0 \end{bmatrix}$$

$$\mathbf{M}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- Top three rows are the linear transform
  - Displacement **d** is properly ignored
- Bottom row stays 0

Primitive Transforms

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_\mathbf{x}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{R}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(\mathbf{d}) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transforms

- Rotation, Scale, and Translation of points and vectors unified in a single matrix transformation:

$$\mathbf{p}' = \mathbf{M}\ \mathbf{p}$$

- Matrix has the form:
  - Last row always 0,0,0,1

$$\begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & d_x \\ m_{yx} & m_{yy} & m_{yz} & d_y \\ m_{zx} & m_{zy} & m_{zz} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transforms compose by matrix multiplication!
  - Same caveat: order of operations is important
  - Same note: Transforms operate right-to-left

Homogeneous coordinates

- Basic: a trick to unify/simplify computations.

- Deeper: projective geometry
  - Interesting mathematical properties
  - Good to know, but less immediately practical
  - We will use some aspect of this when we do perspective projection (in a few weeks)