1. A **definite integral**

$$\int_a^b f(x)\,dx$$

means "find the *signed area* trapped between the graph of f and the interval $[a, b]$."

2. Here is a way to "read" a definite integral.

$$\underbrace{\underbrace{\int_a^b}_{\text{sum}}\ \underbrace{\underbrace{f(x)}_{\text{height}}\ \underbrace{dx}_{\text{width}}}_{\text{area}}}_{\text{total area}}$$

Start with $f(x)$ which represents the height of the function f at the point x. The dx represents a small piece of length located at the point x. Then $f(x)\,dx$ represents a piece of area at the point x, that is $height \times width = area$. Finally, we "sum over" all of the little pieces of length that make up the interval $[a, b]$ to get the "total area," $\int_a^b f(x)\,dx$. (The integral sign, $\int$, is an elongated S and represents the verb "sum." The definite integral, $\int_a^b$, represents summing over the whole row of small rectangular areas that make up the region under the graph of f between the points a and b.)

3. **Linearity of the definite integral.** The integral of a sum of two functions is the sum of two integrals.

$$\int_a^b [\, f(x) + g(x)\,]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

And the integral of a constant times a function is the constant times the integral (or, constants can factor out in front of the integral sign).

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

4. **Additivity** of the definite integral

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

NOTE: In this formula, the point b **need not** be between the points a and c. Draw a picture to see what this formula says when c isn't between a and c.

5. Two more useful properties

$$\int_a^a f(x)\,dx = 0,$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

6. The definite integral is defined, using an infinite limit of **Riemann sums**, as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where the Riemann sum

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

means divide the interval between a and b into n equal size subintervals, each of length $\Delta x = \frac{b-a}{n}$, and then find a value $f(x_i)$ of the function $f(x)$ over the ith subinterval. Each product $f(x_i)\,\Delta x$ represents *the height* $\times$ *the width* of a rectangle, i.e., the *area* of a rectangle. So the sum $\sum_{i=1}^{n} f(x_i)\,\Delta x$ is a sum of many rectangular areas, so it represents an *approximation* of the exact area under the graph of $f(x)$ between a and b. (Question: Why can this “area” be negative?) As we let the number of rectangles get larger (i.e., as $n \to \infty$), the rectangles must get narrower and narrower, and thus the approximate area becomes a better and better approximation of the exact area.

7. The **Fundamental Theorem of Calculus** is

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F'(x) = f(x)$, that is, $F(x)$ is an *antiderivative* of $f(x)$. (Notice that the Fundamental Theorem shows us how the two *different* ideas of “definite” and “indefinite” integrals are connected.)

8. The Fundamental Theorem gives us a reasonable way of computing the *exact* value of a definite integral, as long as we can find the antiderivative F. But finding antiderivatives is hard, so in reality, many calculations of definite integrals are actually done as approximations, on a computer, using many rectangles (or, better yet, trapezoids).

9. The substitution method is done a bit differently with definite integrals. The main thing to remember is to *change the limits of integration when you make the substitution.* Here is a summary of what a substitution looks like with a definite integral.

$$\underbrace{\int_a^b f(g(x))\,g'(x)\,dx}_{\text{start with this problem}} = \underbrace{\int_{g(a)}^{g(b)} f(u)\,du}_{\text{make the } u\text{-substitution}} = \underbrace{F(u)\Big|_{g(a)}^{g(b)}}_{\text{solve the easier problem}} = \underbrace{F(g(b)) - F(g(a))}_{\text{plug in the (new) limits}}$$

In practice, what this means is that in a definite integral, once you replace $g(x)$ with u, you never go back to the x variable.