

Besides the material in this review, be sure to also go over the material from the three midterm exam reviews.

1. In **polar coordinates**,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad dA = r \, dr \, d\theta.$$

2. In **cylindrical coordinates**,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad \text{and} \quad dV = r \, dz \, dr \, d\theta.$$

3. In **spherical coordinates**,

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi, \quad \text{and} \quad dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

4. Given a triple integral of a function $f(x, y, z)$ over a volume V , we can convert the triple integral from rectangular coordinates to cylindrical coordinates,

$$\iiint_V f(x, y, z) \, dV = \iiint_V f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta$$

We can also convert the triple integral from rectangular coordinates to spherical coordinates,

$$\iiint_V f(x, y, z) \, dV = \iiint_V f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

Notice that each of the integrals on the right hand sides is still a triple integral (in cylindrical or spherical coordinates), not yet an iterated integral. To make each triple integral into an iterated integral, you need to choose an order for the iterated integrals and then determine limits of integration for each iterated integral, using the chosen coordinate system, that, in the chosen order, describe the given volume V .

5. Let $\nabla \equiv \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$. We think of the symbol ∇ as representing a certain kind of vector, but it is not really a vector (it is a very convenient notation for a “differential operator”).
6. For a scalar field $f(x, y, z)$ and a vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$,

$$\text{grad}(f) = \nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$$

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = M_x + N_y + P_z$$

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}.$$

7. Notice that ∇f can be thought of as a vector ∇ times a scalar f which results in a vector (the gradient vector field). And $\nabla \cdot \mathbf{F}$ can be thought of as a vector dotted with a vector, so the result should be a scalar (the divergence scalar field). And $\nabla \times \mathbf{F}$ can be thought of as the cross product of two vectors, so the result should be a vector (the curl vector field).
8. If a vector field is only 2-dimensional, $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\text{curl}(\mathbf{F}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

9. The **line integral of a scalar field over a curve** C is defined as

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

where the curve C is parameterized by $x(t)$ and $y(t)$ with $a \leq t \leq b$.

10. The **line integral of a vector field over a curve** C is defined as

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C M(x, y) dx + N(x, y) dy = \int_a^b M(x(t), y(t)) x'(t) + N(x(t), y(t)) y'(t) dt$$

where the curve C is parameterized by $x(t)$ and $y(t)$ with $a \leq t \leq b$.

11. The line integral of a scalar function $f(x, y)$ over a curve C in the xy -plane can be interpreted as the (signed) area of a vertical “curtain” stretching from C up to the graph of f .

If the scalar function is $f(x, y) = 1$, then the scalar line integral of f over a curve C is the arc length of C .

If the scalar function $f(x, y)$ is a density function (usually written as $\delta(x, y)$), then the scalar line integral of $\delta(x, y)$ over a curve C represents the total mass of a wire shaped like C with the variable density at each point in the wire being given by δ .

12. If a vector field \mathbf{F} is interpreted as a force field, then the vector line integral of \mathbf{F} over a curve C represents the work done to move a particle along the curve C from one end of the curve to the other end. To see how this interpretation works, we divide C up into many small pieces of length ds , where ds is small enough so that we can think of each piece of C as being a straight line segment. The work done when moving a particle along one line segment is, as always, *force* \times *distance*. The distance here is ds , the length of the line segment, and the force needs to be the *component of \mathbf{F} that is in the direction of the line segment* (you should draw yourself a small picture). The unit tangent vector \mathbf{T} points in the same direction as the line segment (because the line segment is supposed to be small enough so that it is straight). So the dot product $\mathbf{F} \cdot \mathbf{T}$ is exactly what we need, the component of force in the direction that we are moving. So, for each small

segment of C , the work done is $force \times distance = (\mathbf{F} \cdot \mathbf{T}) ds$. Then the integral sign tells us to add up all the small elements or work to get the total work done while moving over the whole curve, $\int_C (\mathbf{F} \cdot \mathbf{T}) ds$.

13. **Fundamental Theorem of Line Integrals:** If a vector field \mathbf{F} is the gradient of a scalar function f , so $\mathbf{F} = \nabla f$, then for any curve C , with parameterization $x(t)$ and $y(t)$ with $a \leq t \leq b$, we have

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = f(x(b), y(b)) - f(x(a), y(a)).$$

That is, in order to evaluate the line integral of \mathbf{F} over C , all we need to do is evaluate the change in the scalar function f at the two end points of C . Such a scalar function f is called a **potential function** for the vector field \mathbf{F} (it is sometimes called a **stem function** for \mathbf{F}).

14. The Fundamental Theorem of Line Integrals is analogous to the Fundamental Theorem of Calculus because they both say that to evaluate the integral of a function, you only need to find the change in something called an antiderivative (an antiderivative is a function whose derivative is the function you are integrating). In the case of the Fundamental Theorem of Line Integrals, the scalar function f is an “antiderivative” for the vector field \mathbf{F} since we “differentiate” f to get \mathbf{F} , $\nabla f = \mathbf{F}$.

The biggest, most important, difference between the two fundamental theorems is that for the Fundamental Theorem of Calculus, if the integrand is a continuous function, then an antiderivative function always exists, but for the Fundamental Theorem of Line Integrals, it is possible that no “antiderivative” function (i.e., potential function) exists for a vector field \mathbf{F} .

15. If a vector field \mathbf{F} has a potential function f , so $\nabla f = \mathbf{F}$, then $\text{curl}(\mathbf{F}) = 0$. But $\text{curl}(\mathbf{F}) = 0$ does **not** mean that \mathbf{F} has a potential function (see below).
16. It is helpful to make the following definitions.

- A vector field \mathbf{F} is a **conservative vector field** if it is the gradient of a scalar function, that is, there is a scalar function f such that

$$\mathbf{F} = \nabla f.$$

- A vector field \mathbf{F} is **independent of path** if, given any two curves C_1 and C_2 that have the same starting point and the same ending point, then

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T} ds.$$

- If C is a closed curve, so it starts and ends at the same point, then the line integral of a vector field \mathbf{F} over the closed curve C is called the **circulation** of \mathbf{F} around C .

17. Let \mathbf{F} be a vector field defined on a region R . Then the following three statements are equivalent, meaning that if one of them is a true statement about \mathbf{F} , then the other two must also be true statements about \mathbf{F}

- \mathbf{F} is a conservative vector field.
- \mathbf{F} is independent of path.
- The circulation of \mathbf{F} around every closed curve is zero.

If, in addition, the region R does not have any holes in it (we say the region is **simply connected**), then we can add a fourth equivalent condition to the above list.

- $\text{curl}(\mathbf{F}) = 0$.

18. When a 2-dimensional vector field $\mathbf{F}(x, y)$ is defined on a simply connected region and the vector field has the property that $\text{curl}(\mathbf{F}(x, y)) = 0$, then there is a procedure that lets us calculate the potential function $f(x, y)$ for $\mathbf{F}(x, y)$.

Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field with the property

$$\text{curl}(\mathbf{F}(x, y)) = \frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} = 0.$$

We want to find a function $f(x, y)$ that solves $\nabla f(x, y) = \mathbf{F}(x, y)$. More specifically, given the two functions $M(x, y)$ and $N(x, y)$, we want to find $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).$$

Step 1: Compute the antiderivative of $M(x, y)$ with respect to x . This should give us, according to the equation just above on the left, $f(x, y)$ plus some unknown constant of integration. But that unknown “constant of integration” is in fact an unknown function of y (since the derivative with respect to x of any function of y must be 0). So Step 1 gives us

$$f(x, y) = \int M(x, y) dx + g(y)$$

where we do not, yet, know what the function $g(y)$ is.

Step 2: Differentiate the answer from Step 1 with respect to y and set the result equal to $N(x, y)$ (since it must be that $f_y(x, y) = N(x, y)$).

$$N(x, y) = \frac{\partial}{\partial y} \left[\int M(x, y) dx + g(y) \right] = \frac{\partial}{\partial y} \left[\int M(x, y) dx \right] + g'(y)$$

Step 3: Solve the equation from the last step for $g'(y)$.

Step 4: Find the antiderivative with respect to y of $g'(y)$,

$$g(y) = \int g'(y) dy.$$

Step 5: Substitute the function $g(y)$ found in Step 4 into the answer from Step 1.

19. The formula from Green's Theorem:

$$\oint_C M(x, y) dx + N(x, y) dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$