1. Given a point (x_0, y_0), the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ has many different slopes, or "steepnesses." In fact, the graph will have a different slope in every different direction. The slope of the graph at a given point and in a given direction is called a **directional derivative**.

2. We use the gradient vector to compute the directional derivative of $f(x, y)$ at a given point (x_0, y_0) in a given direction $\mathbf{u}$ (where $\mathbf{u}$ must be a unit vector, $||\mathbf{u}|| = 1$),

$$D_{\mathbf{u}} f(x_0, y_0) = \mathbf{u} \cdot \nabla f(x_0, y_0).$$

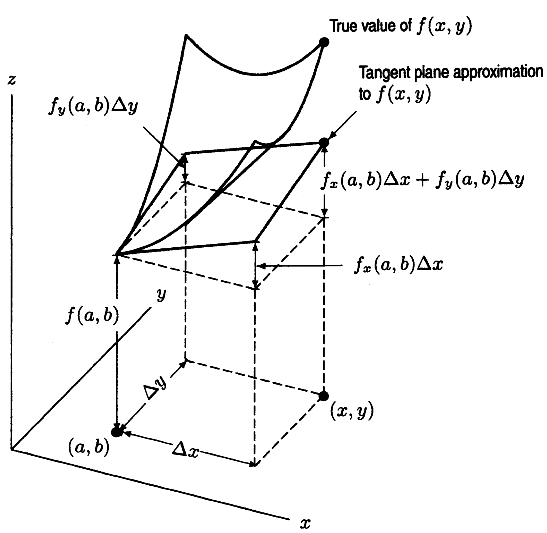
   Notice how we are using unit vectors to describe direction.

3. Here is a way to "derive" the above formula for the directional derivative. Start with the equation for the tangent plane.

$$\begin{aligned}
T(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
&= f(x_0, y_0) + (f_x(x_0, y_0),\ f_y(x_0, y_0)) \cdot (x - x_0,\ y - y_0) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot \left(\frac{\mathbf{x} - \mathbf{x_0}}{||\mathbf{x} - \mathbf{x_0}||}\, ||\mathbf{x} - \mathbf{x_0}|| \right) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{u}\, ||\mathbf{x} - \mathbf{x_0}||) \quad (\mathbf{u} = \text{unit vector in the direction of } \mathbf{x} - \mathbf{x_0}) \\
&= f(\mathbf{x_0}) + (\mathbf{u} \cdot \nabla f(\mathbf{x_0}))\ ||\mathbf{x} - \mathbf{x_0}|| \quad (\text{algebraic properties of the dot product}) \\
&= f(\mathbf{x_0}) + \underbrace{\underbrace{D_{\mathbf{u}} f(\mathbf{x_0})}_{\text{slope}}\ \underbrace{||\mathbf{x} - \mathbf{x_0}||}_{\text{run}}}_{\text{rise} = \text{slope} \times \text{run}} \quad (\text{definition of the directional derivative})
\end{aligned}$$

   Notice how the last equation reads almost exactly like the tangent line equation from first semester calculus.

$$T(x) = f(x_0) + \underbrace{\underbrace{f'(x_0)}_{\text{slope}}\ \underbrace{(x - x_0)}_{\text{run}}}_{\text{rise} = \text{slope} \times \text{run}}$$



4. Given a point (x_0, y_0), the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ has many different slopes, or "steepnesses." The maximal slope of the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is in the direction of the gradient vector. Another way to put this is that the maximal directional derivative is in the direction of the gradient. Also, the maximal slope is equal to the length of the gradient,

$$D_{\mathbf{u}_{\nabla f}} f(x_0, y_0) = ||\nabla f(x_0, y_0)||,$$

where $\mathbf{u}_{\nabla f}$ is the unit vector in the direction of the gradient. The reason for these two facts is that for any direction $\mathbf{u}$,

$$D_{\mathbf{u}} f(x_0, y_0) = \mathbf{u} \cdot \nabla f(x_0, y_0) = ||\mathbf{u}|| \, ||\nabla f(x_0, y_0)|| \cos\theta = ||\nabla f(x_0, y_0)|| \cos\theta \le ||\nabla f(x_0, y_0)||.$$

So the most that $D_{\mathbf{u}} f(x_0, y_0)$ can be is $||\nabla f(x_0, y_0)||$, and $D_{\mathbf{u}} f(x_0, y_0) = ||\nabla f(x_0, y_0)||$ exactly when the angle between $\mathbf{u}$ and $\nabla f(x_0, y_0)$ is zero (that is, $\mathbf{u}$ points in the direction of $\nabla f(x_0, y_0)$).

5. Given a point (x_0, y_0), if we compute $c = f(x_0, y_0)$, then the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve of height c. (Remember that level curves and the gradient vector are two-dimensional objects; they live in the two-dimensional xy-plane, not in the three-dimensional xyz-space.) Be sure to compare this fact with the previous item. The gradient points in the direction of steepest increase and the negative of the gradient points in the direction of steepest decrease. This item says that the direction of "no increase or decrease" is exactly "half way" between the steepest increase and the steepest decrease.

6. Remember that partial derivatives are also directional derivatives,

$$f_x(x, y) = D_{\mathbf{i}} f(x, y) \qquad \text{and} \qquad f_y(x, y) = D_{\mathbf{j}} f(x, y)$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

7. We have two ways of describing surfaces in three-dimensional xyz-space. We can describe a surface as the graph of a function of two variables, $z = f(x, y)$. Or we can describe a surface as a level set of a function of three variables, $c = F(x, y, z)$, where c is a given constant. You need to know what the normal vector to a point (x_0, y_0, z_0) on the surface is in each case. For the first case, the normal vector is as

$$\mathbf{n} = (\, -f_x(x_0, y_0), \; -f_y(x_0, y_0), \; 1 \,).$$

For the second case, since a gradient vector is always perpendicular to its level set, the normal vector is the gradient vector of the function F,

$$\mathbf{n} = \nabla F(x_0, y_0, z_0) = (\, F_x(x_0, y_0, z_0), \; F_y(x_0, y_0, z_0), \; F_z(x_0, y_0, z_0) \,).$$