1. For a function of two variables $f(x, y)$, a **vertical slice** is the intersection of the graph of $z = f(x, y)$ with a plane perpendicular to the xy-plane. Know how to use the idea of a vertical slice to describe what we mean by the "partial derivatives" of $f(x, y)$ at a given point (x_0, y_0).

2. For a function of two variables $f(x, y)$ and a point (x_0, y_0), the two vectors

$$(\,1,\, 0,\, \frac{\partial f}{\partial x}(x_0, y_0)\,) \quad \text{and} \quad (\,0,\, 1,\, \frac{\partial f}{\partial y}(x_0, y_0)\,)$$

are both tangent vectors to the graph of $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

3. The cross product of the two vectors from the previous item,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} = (\,-\frac{\partial f}{\partial x}(x_0, y_0),\, -\frac{\partial f}{\partial y}(x_0, y_0),\, 1\,)$$

is perpendicular to the graph of $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$, so it is a normal vector for the plane tangent to the graph of $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

4. Let $\mathbf{x_0} = (x_0, y_0, f(x_0, y_0))$ be a given point on the graph of $z = f(x, y)$, let $\mathbf{n} = (\,-f_x(x_0, y_0),\, -f_y(x_0, y_0),\, 1\,)$ be the normal vector at that point, and let $\mathbf{x} = (x, y, z)$ be a point on the plane tangent to the graph of $z = f(x, y)$ at the given point. Then the vector equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) = 0$ for this tangent plane can be rewritten as

$$(\,-f_x(x_0, y_0),\, -f_y(x_0, y_0),\, 1\,) \cdot (\,x - x_0,\, y - y_0,\, z - f(x_0, y_0)\,) = 0.$$

Doing the dot product, we get

$$-f_x(x_0, y_0)\,(x - x_0) - f_y(x_0, y_0)\,(y - y_0) + z - f(x_0, y_0) = 0.$$

Rearranging terms, we get the following equation for the tangent plane (be sure to compare this equation with the picture on the next page).

$$z = f(x_0, y_0) + \underbrace{\underbrace{f_x(x_0, y_0)}_{\text{slope}}\,\underbrace{(x - x_0)}_{\text{run}}}_{\text{rise in the } x\text{-direction}} + \underbrace{\underbrace{f_y(x_0, y_0)}_{\text{slope}}\,\underbrace{(y - y_0)}_{\text{run}}}_{\text{rise in the } y\text{-direction}}.$$

Notice its similarity to the equation for a tangent *line* at the point $(x_0, f(x_0))$ for a function of *one* variable $y = f(x)$,

$$y = f(x_0) + \underbrace{\underbrace{f'(x_0)}_{\text{slope}}\,\underbrace{(x - x_0)}_{\text{run}}}_{\text{rise = slope} \times \text{run}}$$

5. The tangent plane formula is itself a function of x and y, $T_{(x_0,y_0)}(x, y)$, that we call the **tangent plane approximation function** for $f(x, y)$ at the point (x_0, y_0),

$$T_{(x_0,y_0)}(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

For arbitrary points (x, y) "near" the given point (x_0, y_0)

$$f(x, y) \approx T_{(x_0,y_0)}(x, y).$$

The tangent plane function approximates the original function.

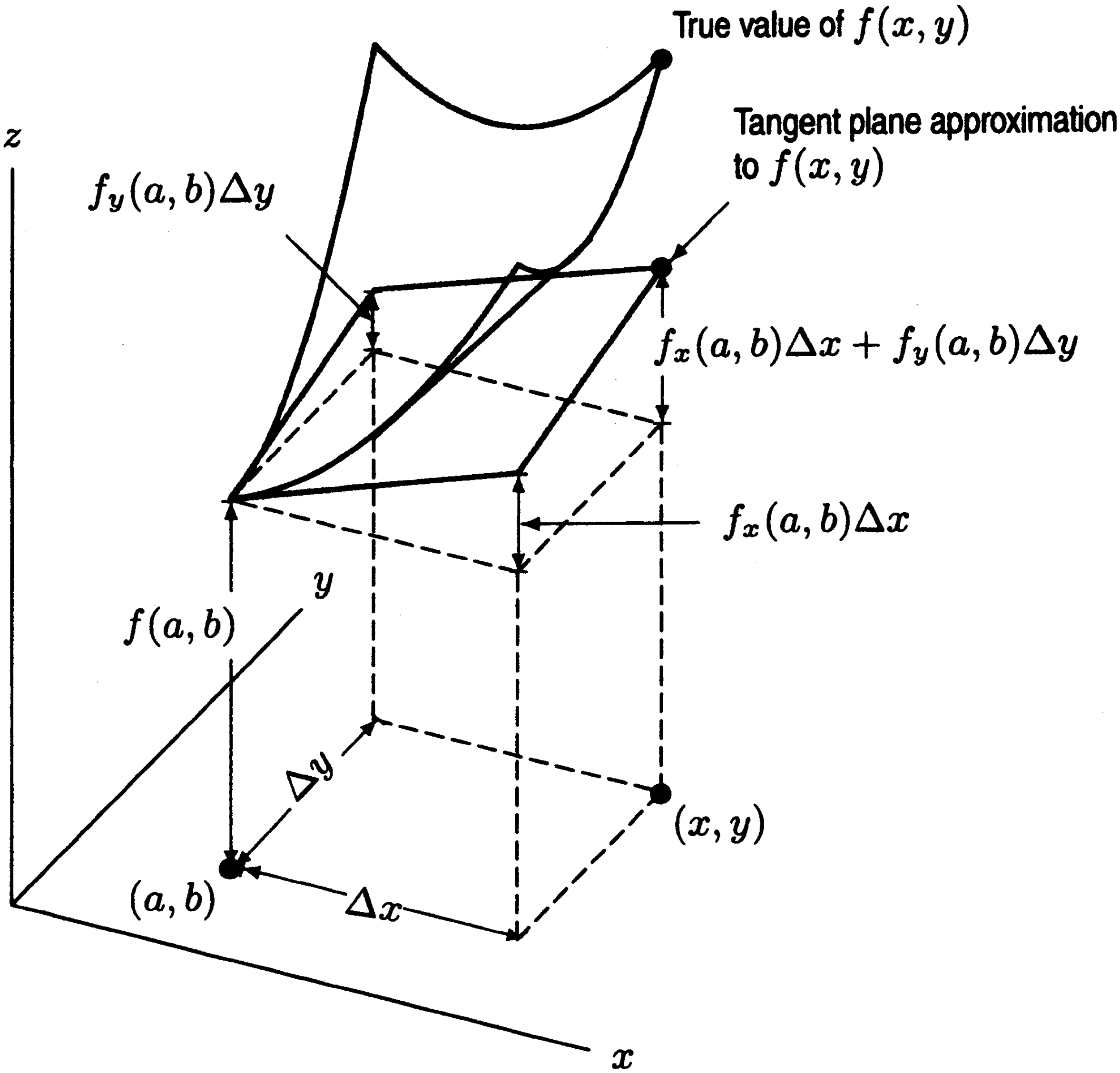

Figure 14.22: Local linearization: Approximating $f(x, y)$ by the z-value from the tangent plane

6. For the function $f(x, y)$, the **gradient vector** is the two-dimensional vector

$$\nabla f(x, y) = (\, f_x(x, y),\ f_y(x, y)\,) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

7. Know the derivative rules for the gradient as given in Theorem B on page 642.

8. Let $\mathbf{x_0} = (x_0, y_0)$ and $\mathbf{x} = (x, y)$ denote a known and an unknown point. We can use the gradient vector to rewrite the formula for the plane tangent to the graph of $f(x, y) = f(\mathbf{x})$ at the point $(x_0, y_0, f(x_0, y_0)) = (\mathbf{x_0}, f(\mathbf{x_0}))$. Start with the formula from item 5.

$$T(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Interpret the right hand side as a dot product of *two-dimensional* vectors.

$$T(x, y) = f(x_0, y_0) + (\, f_x(x_0, y_0),\ f_y(x_0, y_0)\,) \cdot (\, x - x_0,\ y - y_0\,)$$

Interpret the second vector in the dot product as a difference of two vectors.

$$T(x, y) = f(x_0, y_0) + (\, f_x(x_0, y_0),\ f_y(x_0, y_0)\,) \cdot (\,(x, y) - (x_0, y_0)\,)$$

Now replace each (x_0, y_0) with $\mathbf{x_0}$ and each (x, y) with $\mathbf{x}$.

$$T(\mathbf{x}) = f(\mathbf{x_0}) + (\, f_x(\mathbf{x_0}),\ f_y(\mathbf{x_0})\,) \cdot (\mathbf{x} - \mathbf{x_0})$$

Now use the definition of the gradient.

$$T(\mathbf{x}) = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})$$

Notice how much this looks like the formula for the tangent *line* at the point $(x_0, f(x_0))$ for a function of *one* variable $f(x)$.

$$T(x) = f(x_0) + f'(x_0)(x - x_0)$$