MA 26100 **Vector Algebra**

Let **u** and **v** be three-dimensional vectors. We have a few different ways to write vectors.

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Here are the algebra rules for vectors.

1. $\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j} + (u_3 + v_3)\mathbf{k} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$
2. $a\mathbf{u} = (au_1)\mathbf{i} + (au_2)\mathbf{j} + (au_3)\mathbf{k} = (au_1,\ au_2,\ au_3)$
3. $||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
4. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative law)
5. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative law)
6. $(ab)\mathbf{u} = a(b\mathbf{u})$ (Associative law)
7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (Distributive law)
8. $(a + b)\mathbf{u} =$ a**u** + b**u** (Distributive law)
9. $||a\mathbf{u}|| = |a|\,||\mathbf{u}||$
10. $\left|\left|\dfrac{\mathbf{v}}{||\mathbf{v}||}\right|\right| = 1$ (The vector $\dfrac{\mathbf{v}}{||\mathbf{v}||}$ is called the *unit vector in the direction of* **v**.)
11. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ (Algebraic definition of the **dot product**.)
12. $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta$ where θ is the angle between **u** and **v**.
13. $\mathbf{u} \cdot \mathbf{v} = 0$ means that **u** and **v** are perpendiccular.
14. $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$ (Another, less usefull, way to say this is $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.)
15. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Commutative law)
16. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Distributive law)
17. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$ (Associative law)
18. $\mathbf{proj_u}\,\mathbf{v} = \left(\dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = \left(\dfrac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\right)\mathbf{u} = \left(\dfrac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||}\right)\dfrac{\mathbf{u}}{||\mathbf{u}||} = \left(\mathbf{v} \cdot \dfrac{\mathbf{u}}{||\mathbf{u}||}\right)\dfrac{\mathbf{u}}{||\mathbf{u}||}.$
19. $\text{comp}_\mathbf{u}\,\mathbf{v} = \mathbf{v} \cdot \dfrac{\mathbf{u}}{||\mathbf{u}||} = ||\mathbf{v}|| \cos\theta$ (This is the *component of* **v** *in the direction of* **u**.)

20. $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ is the equation of the plane containing the given point $\mathbf{p} = (x_1, y_1, z_1)$ and the given normal vector $\mathbf{n} = (A, B, C)$. Another way to write this is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

or even

$$Ax + By + Cz = D \quad \text{where } D = Ax_1 + By_1 + Cz_1.$$

21. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ (Algebraic definition of the **cross product**.)

22. $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}|| \sin\theta =$ the area of the parallelogram determined by **u** and **v**.

23. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ means that **u** and **v** are parallel.

24. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ (The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both **u** and **v**.)

25. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (Anticommutativity)

26. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ (Left distributive law)

27. $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (a\mathbf{v})$ (Associative law)

28. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

29. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ (Notice that $\mathbf{0}$ is the zero *vector*, not the number zero.)

30. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ (The "scalar triple product".)

31. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$ the volume of the parallelepiped determined by **u**, **v**, and **w**

32. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

33. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (The "vector triple product".)

34. The vector triple product is often written the following way,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

and it is remembered by using the mnemonic "bac-cab".

35. **NOTE:** The cross product is **not** associative. That is,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

36. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$