1. Find the eigenvalues (but not the eigenvectors) for the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. Show your work.

Solution: We need to find the solutions of the charateristic equation.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 1-\lambda & 2 \\ 4 & 8-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(8-\lambda) - 8 &= 0 \\ (\lambda-1)(\lambda-8) - 8 &= 0 \\ \lambda^2 - 9\lambda + 8 - 8 &= 0 \\ \lambda^2 - 9\lambda &= 0 \\ \lambda(\lambda - 9) &= 0 \end{aligned}$$

So the eigenvalues are $\lambda = 0$ or $\lambda = 9$.

2. Find matrices P and D, where D is a diagonal matrix, such that $A = PDP^{-1}$. Show your work.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Solution: Since the matrix is triangular, the eigenvalues are the entries on the main diagonal. So the eigenvalues are $\lambda = 4,\ 3,\ 2$. For each eigenvalue we need to compute an eigenvector.

For $\lambda = 4$ we need to solve $(A - 4I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = 2t$, $x_2 = 2t$, and $x_3 = t$. The eigenvector is

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

For $\lambda = 3$ we need to solve $(A - 3I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = 0$, $x_2 = t$, and $x_3 = 0$. The eigenvector is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 2$ we need to solve $(A - 2I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = 0$, $x_2 = 0$, and $x_3 = t$. The eigenvector is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the values for D and P are $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$