

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
 - **Representation: unsigned and signed**
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - Summary
- Representations in memory, pointers, strings
- Summary

Encoding Integers

Unsigned

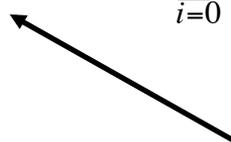
$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

```
short int x = 15213;
short int y = -15213;
```

Sign
Bit



■ C short 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

■ Sign Bit

- For 2's complement, most significant bit indicates sign
 - 0 for nonnegative
 - 1 for negative

Two-complement Encoding Example (Cont.)

```

x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
  
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		15213		-15213

Numeric Ranges

■ Unsigned Values

- $UMin = 0$
000...0
- $UMax = 2^w - 1$
111...1

■ Two's Complement Values

- $TMin = -2^{w-1}$
100...0
- $TMax = 2^{w-1} - 1$
011...1

■ Other Values

- Minus 1
111...1

Values for $W = 16$

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

■ Observations

- $|TMin| = TMax + 1$
 - Asymmetric range
- $UMax = 2 * TMax + 1$

■ C Programming

- `#include <limits.h>`
- Declares constants, e.g.,
 - `ULONG_MAX`
 - `LONG_MAX`
 - `LONG_MIN`
- Values platform specific

Unsigned & Signed Numeric Values

X	B2U(X)	B2T(X)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

■ Equivalence

- Same encodings for nonnegative values

■ Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

■ ⇒ Can Invert Mappings

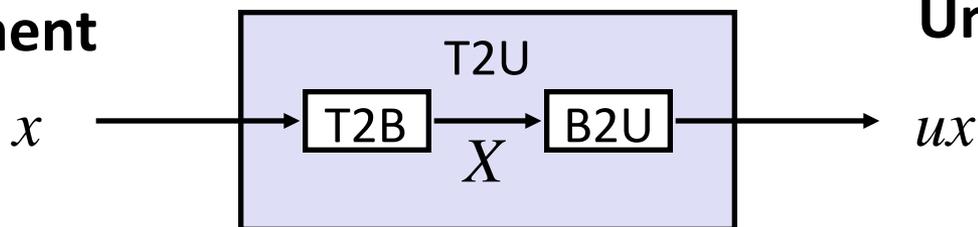
- $U2B(x) = B2U^{-1}(x)$
 - Bit pattern for unsigned integer
- $T2B(x) = B2T^{-1}(x)$
 - Bit pattern for two's comp integer

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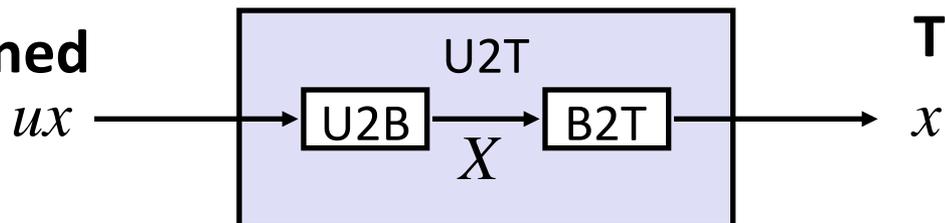
Mapping Between Signed & Unsigned

Two's Complement



Maintain Same Bit Pattern

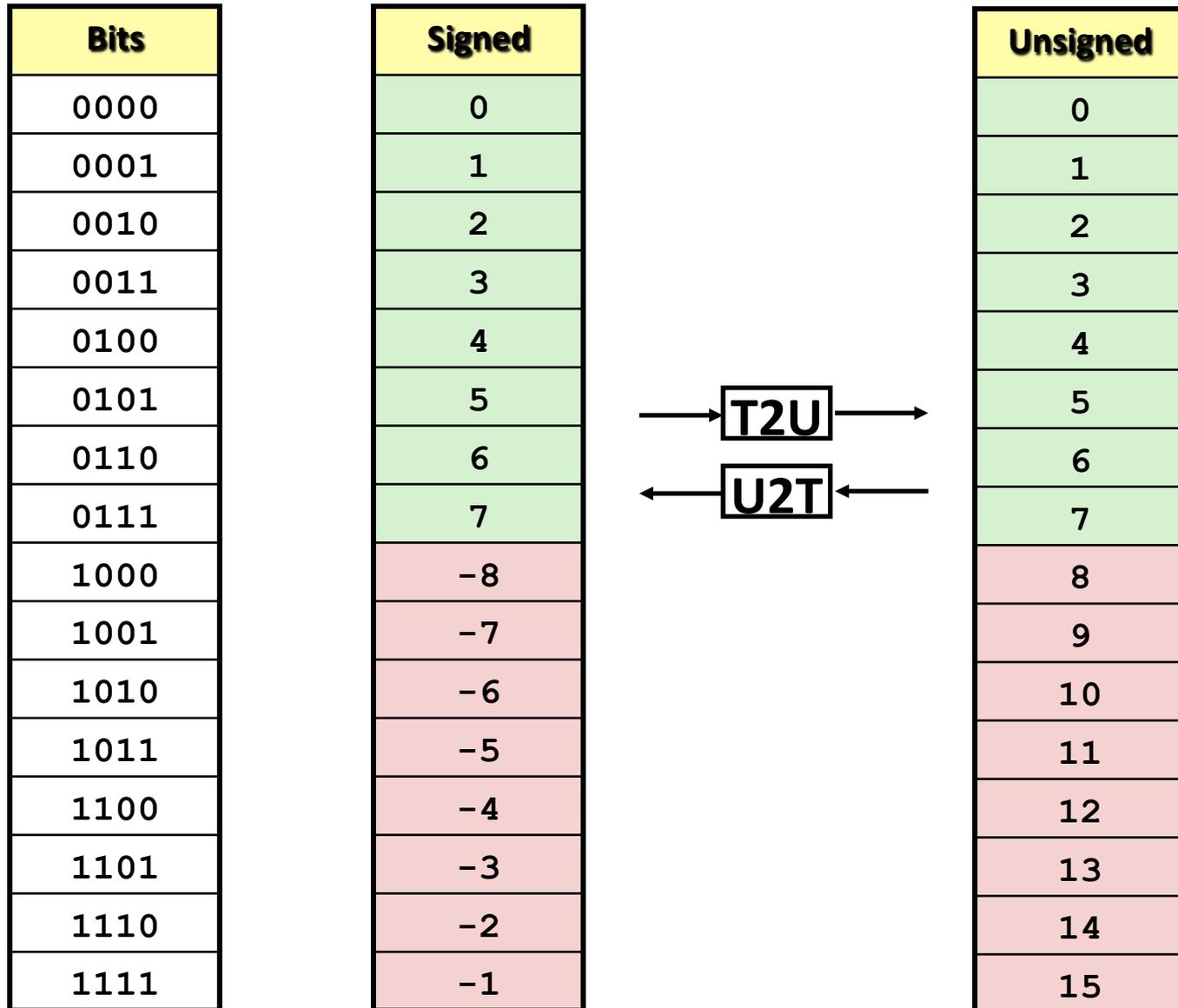
Unsigned



Maintain Same Bit Pattern

- Mappings between unsigned and two's complement numbers:
Keep bit representations and reinterpret

Mapping Signed \leftrightarrow Unsigned

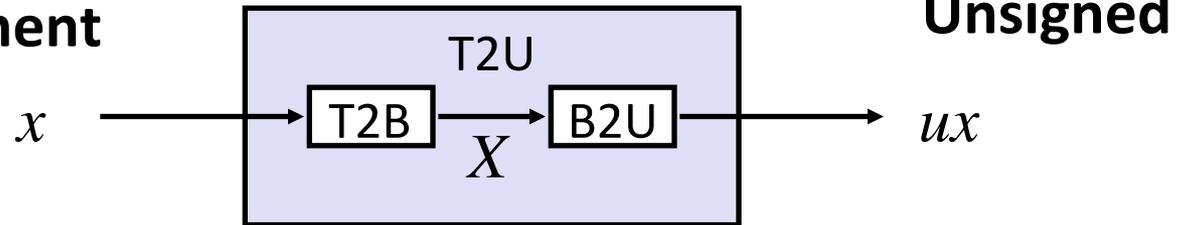


Mapping Signed \leftrightarrow Unsigned

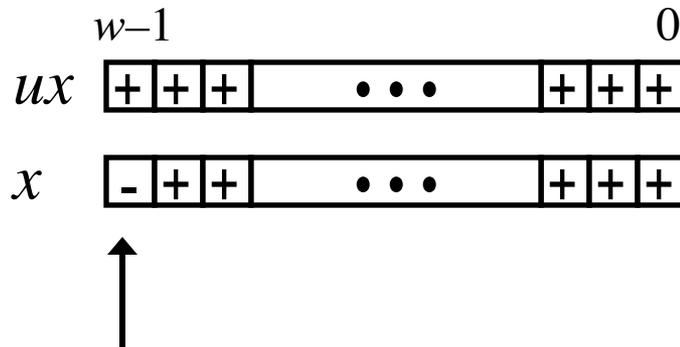
Bits	Signed		Unsigned
0000	0	=	0
0001	1		1
0010	2		2
0011	3		3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8	+/- 16	8
1001	-7		9
1010	-6		10
1011	-5		11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Relation between Signed & Unsigned

Two's Complement



Maintain Same Bit Pattern



Large negative weight

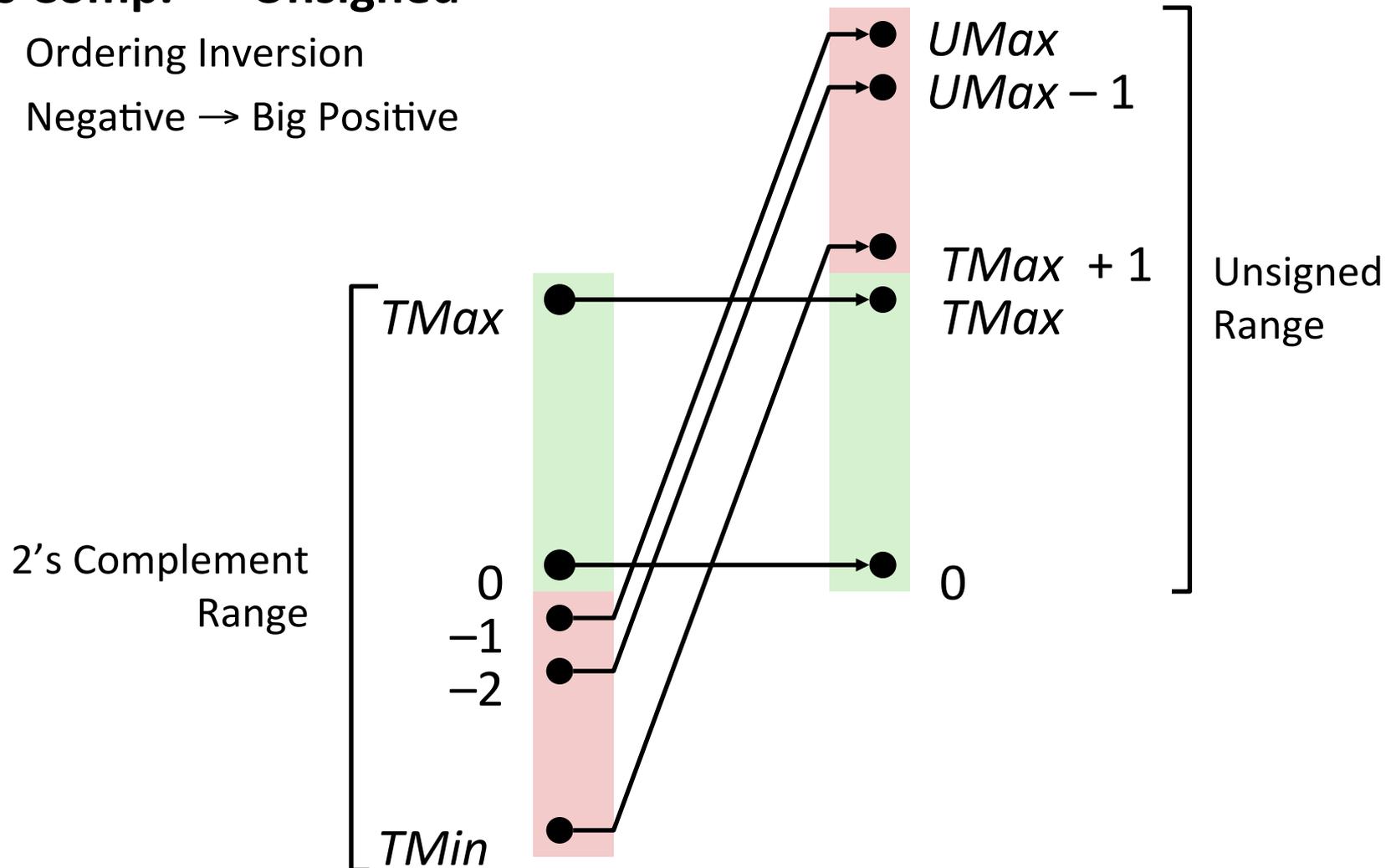
becomes

Large positive weight

Conversion Visualized

■ 2's Comp. → Unsigned

- Ordering Inversion
- Negative → Big Positive



Signed vs. Unsigned in C

■ Constants

- By default are considered to be signed integers
- Unsigned if have “U” as suffix

`0U, 4294967259U`

■ Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

Casting Surprises

■ Expression Evaluation

- If there is a mix of unsigned and signed in single expression, *signed values implicitly cast to unsigned*
- Including comparison operations $<$, $>$, $==$, $<=$, $>=$
- Examples for $W = 32$: **TMIN = -2,147,483,648** , **TMAX = 2,147,483,647**

■ Constant ₁	Constant ₂	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Summary

Casting Signed \leftrightarrow Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w

- Expression containing signed and unsigned int
 - `int` is cast to `unsigned`!!

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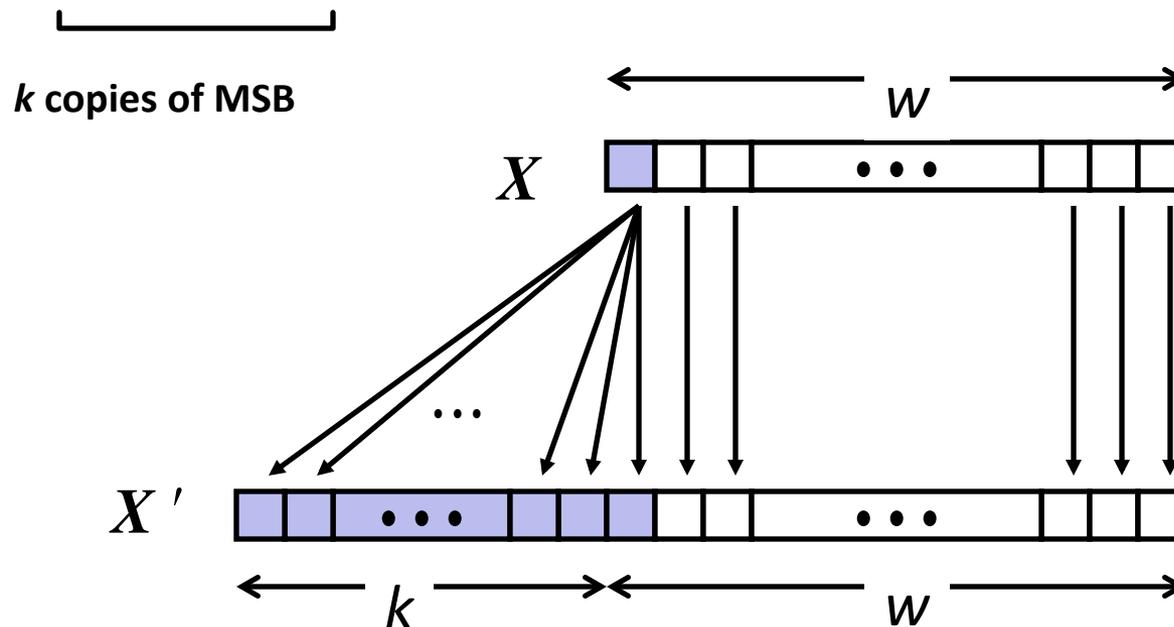
Sign Extension

■ Task:

- Given w -bit signed integer x
- Convert it to $w+k$ -bit integer with same value

■ Rule:

- Make k copies of sign bit:
- $X' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \dots, x_0$



Sign Extension Example

```
short int x = 15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary:

Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
 - Unsigned: zeros added
 - Signed: sign extension
 - Both yield expected result

- **Truncating (e.g., unsigned to unsigned short)**
 - Unsigned/signed: bits are truncated
 - Result reinterpreted
 - Unsigned: mod operation
 - Signed: similar to mod
 - For small numbers yields expected behavior

Today: Bits, Bytes, and Integers

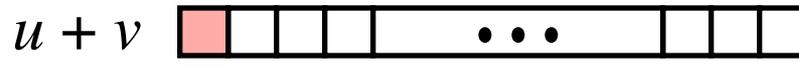
- Representing information as bits
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
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Unsigned Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ Standard Addition Function

- Ignores carry output

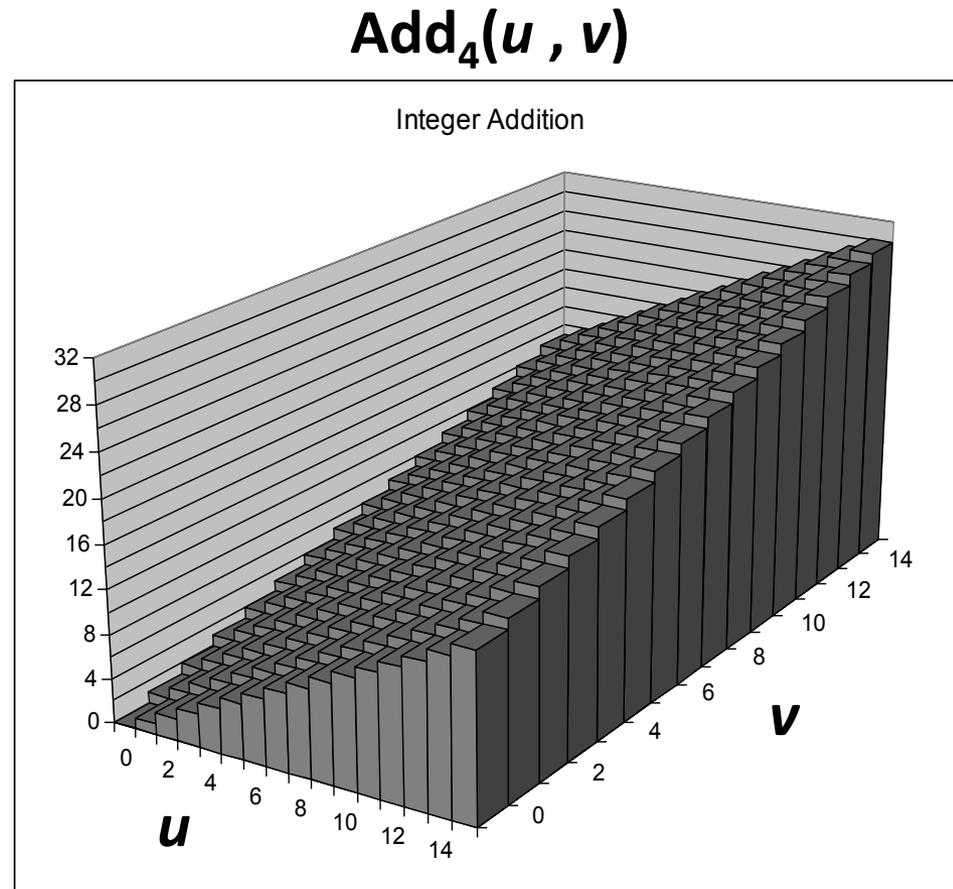
■ Implements Modular Arithmetic

$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

Visualizing (Mathematical) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface



Visualizing Unsigned Addition

■ Wraps Around

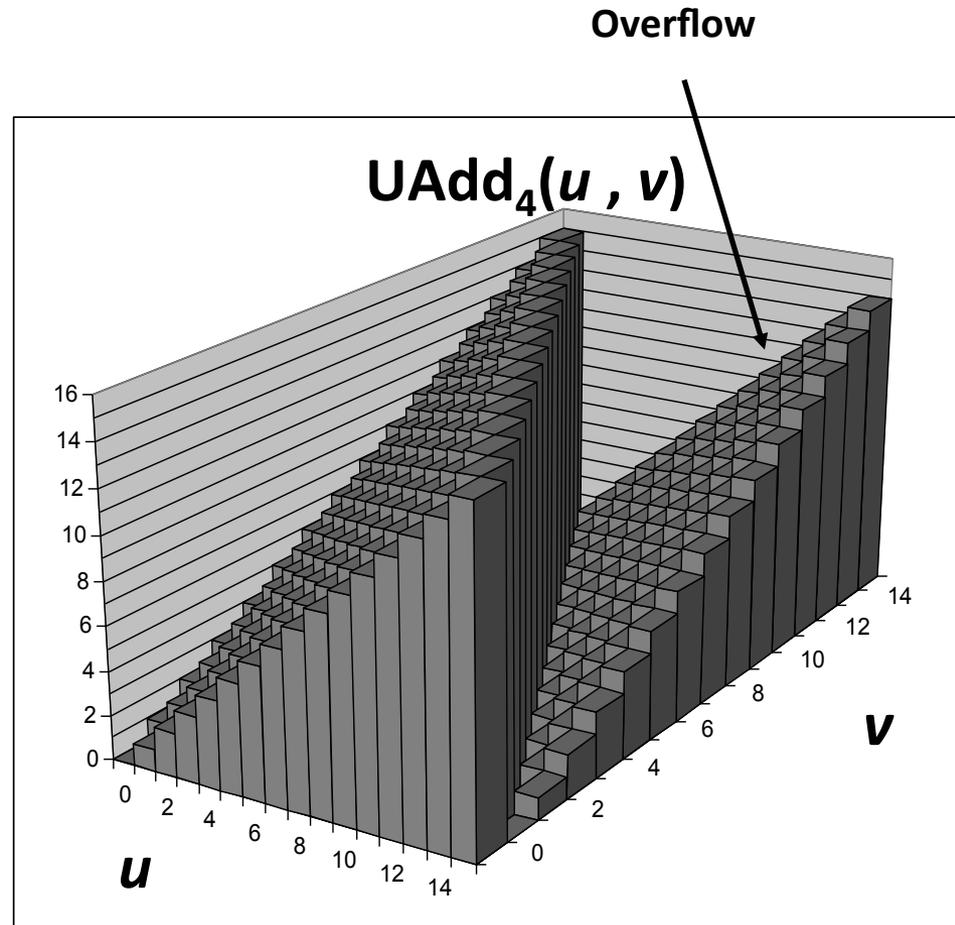
- If true sum $\geq 2^w$
- At most once

True Sum

2^{w+1}
 2^w
 0

Overflow

Modular Sum



Two's Complement Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

```
int s, t, u, v;
```

```
s = (int) ((unsigned) u + (unsigned) v);
```

```
t = u + v
```

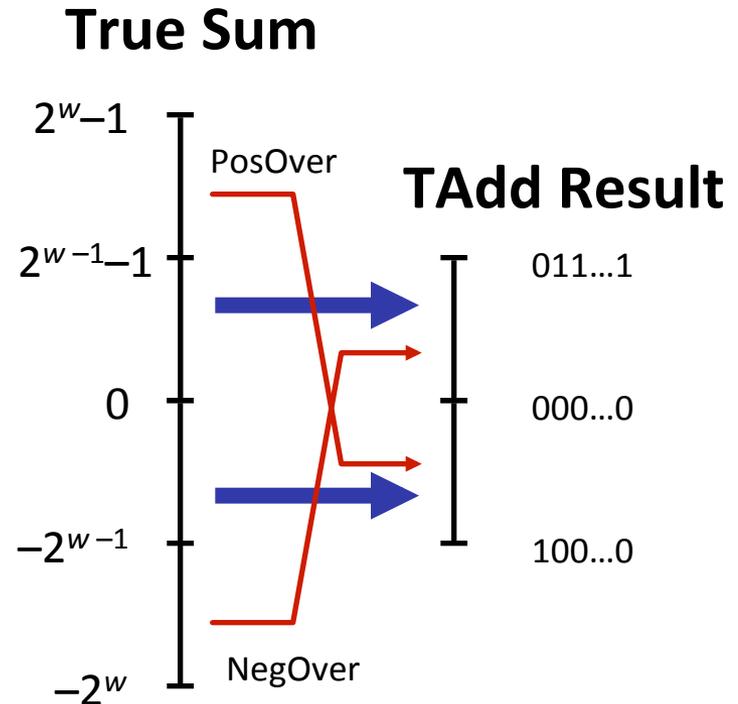
- Will give `s == t`

TAdd Overflow

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer

0 111...1
 0 100...0
 0 000...0
 1 011...1
 1 000...0



Visualizing 2's Complement Addition

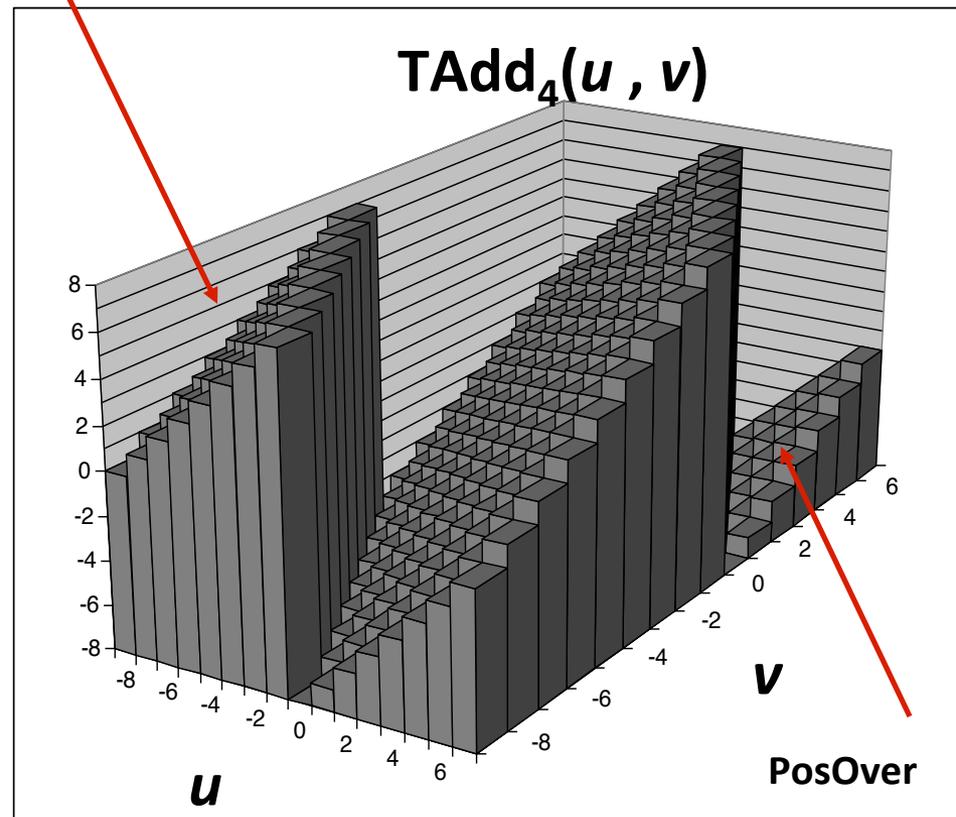
■ Values

- 4-bit two's comp.
- Range from -8 to +7

■ Wraps Around

- If $\text{sum} \geq 2^{w-1}$
 - Becomes negative
 - At most once
- If $\text{sum} < -2^{w-1}$
 - Becomes positive
 - At most once

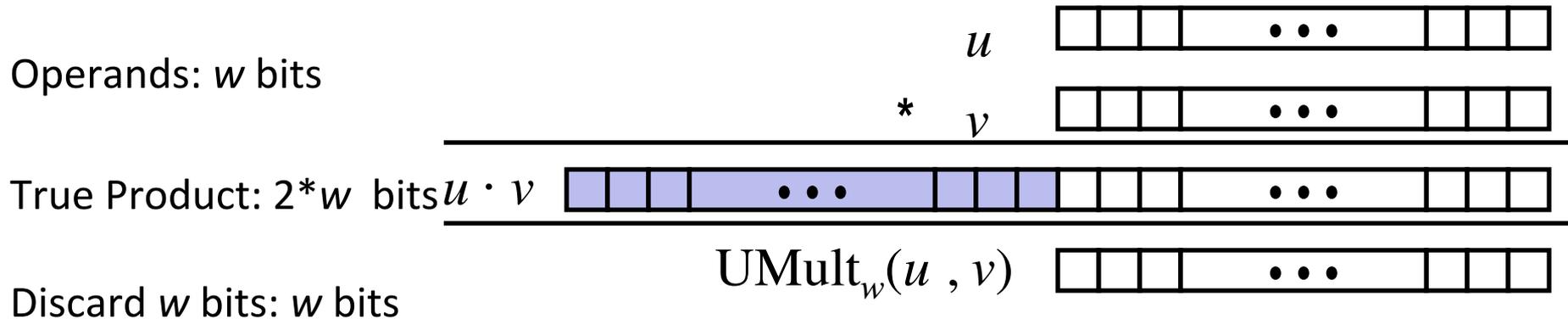
NegOver



Multiplication

- **Goal: Computing Product of w -bit numbers x, y**
 - Either signed or unsigned
- **But, exact results can be bigger than w bits**
 - Unsigned: up to $2w$ bits
 - Result range: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
 - Two's complement min (negative): Up to $2w-1$ bits
 - Result range: $x * y \geq (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to $2w$ bits, but only for $(TMin_w)^2$
 - Result range: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
- **So, maintaining exact results...**
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by “arbitrary precision” arithmetic packages

Unsigned Multiplication in C



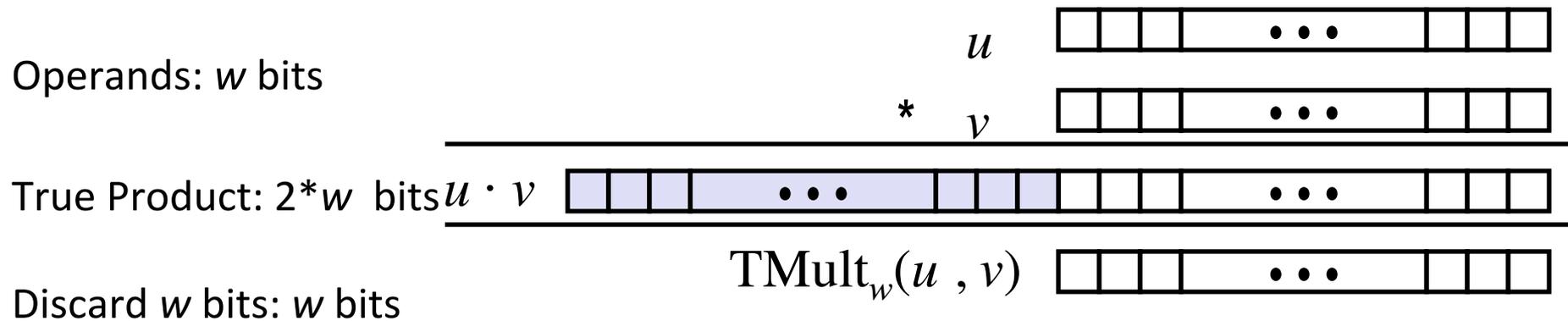
■ Standard Multiplication Function

- Ignores high order w bits

■ Implements Modular Arithmetic

$$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$$

Signed Multiplication in C



■ Standard Multiplication Function

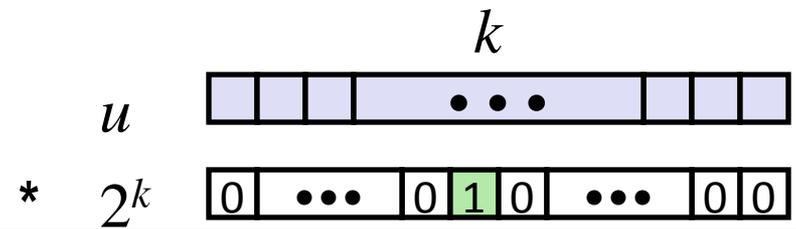
- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

Power-of-2 Multiply with Shift

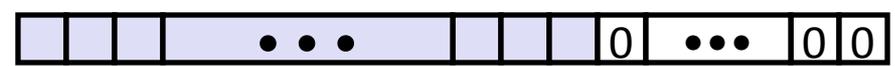
■ Operation

- $u \ll k$ gives $u * 2^k$
- Both signed and unsigned

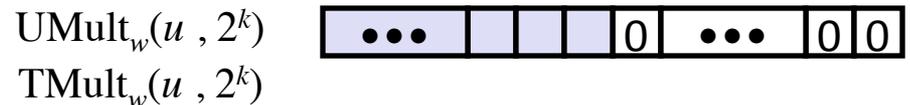
Operands: w bits



True Product: $w+k$ bits $u \cdot 2^k$



Discard k bits: w bits



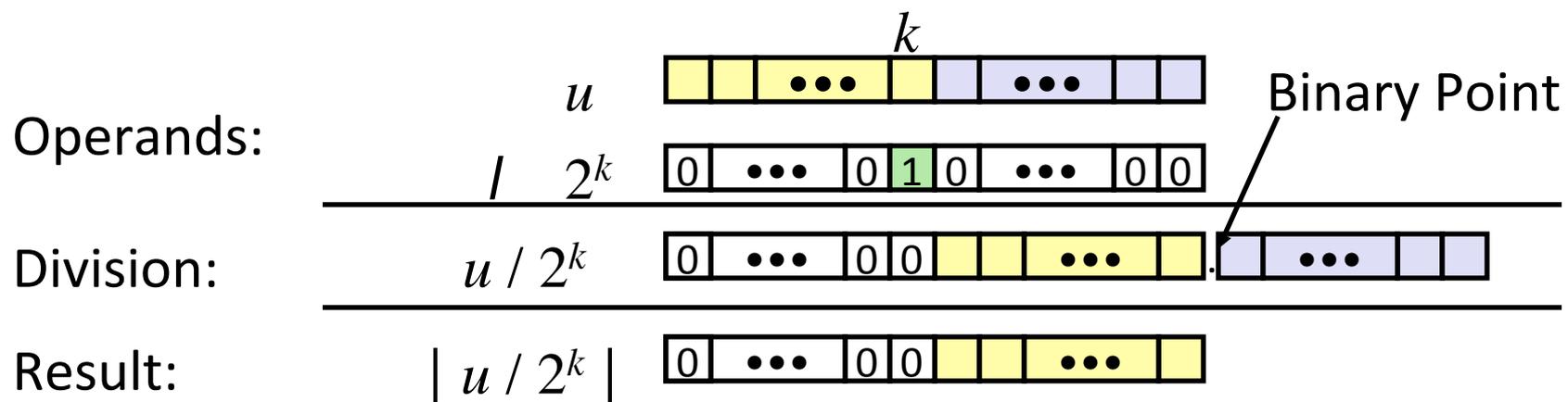
■ Examples

- $u \ll 3 \quad \quad \quad == \quad u * 8$
- $(u \ll 5) - (u \ll 3) == \quad u * 24$
- Most machines shift and add faster than multiply
 - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

■ Quotient of Unsigned by Power of 2

- $u \gg k$ gives $\lfloor u / 2^k \rfloor$
- Uses logical shift



	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	00011101 10110110
x >> 4	950.8125	950	03 B6	00000011 10110110
x >> 8	59.4257813	59	00 3B	00000000 00111011

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Arithmetic: Basic Rules

■ Addition:

- Unsigned/signed: Normal addition followed by truncate, same operation on bit level
- Unsigned: addition mod 2^w
 - Mathematical addition + possible subtraction of 2^w
- Signed: modified addition mod 2^w (result in proper range)
 - Mathematical addition + possible addition or subtraction of 2^w

■ Multiplication:

- Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
- Unsigned: multiplication mod 2^w
- Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

■ *Don't* use without understanding implications

- Easy to make mistakes

```
unsigned i;  
for (i = cnt-2; i >= 0; i--)  
    a[i] += a[i+1];
```

- Can be very subtle

```
#define DELTA sizeof(int)  
int i;  
for (i = CNT; i-DELTA >= 0; i-= DELTA)  
    . . .
```

Counting Down with Unsigned

■ Proper way to use unsigned as loop index

```
unsigned i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

■ See Robert Seacord, *Secure Coding in C and C++*

- C Standard guarantees that unsigned addition will behave like modular arithmetic
 - $0 - 1 \rightarrow UMax$

■ Even better

```
size_t i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

- Data type `size_t` defined as unsigned value with length = word size
- Code will work even if `cnt = UMax`
- What if `cnt` is signed and < 0 ?

Why Should I Use Unsigned? (cont.)

- **Do Use When Performing Modular Arithmetic**
 - Multiprecision arithmetic
- **Do Use When Using Bits to Represent Sets**
 - Logical right shift, no sign extension

Integer C Puzzles

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- $x < 0 \Rightarrow ((x*2) < 0)$
- $ux \geq 0$
- $x \& 7 == 7 \Rightarrow (x \ll 30) < 0$
- $ux > -1$
- $x > y \Rightarrow -x < -y$
- $x * x \geq 0$
- $x > 0 \ \&\& \ y > 0 \Rightarrow x + y > 0$
- $x \geq 0 \Rightarrow -x \leq 0$
- $x \leq 0 \Rightarrow -x \geq 0$
- $(x | -x) \gg 31 == -1$
- $ux \gg 3 == ux / 8$
- $x \gg 3 == x / 8$
- $x \& (x-1) != 0$

Bonus extras

Binary Number Property

Claim

$$1 + 1 + 2 + 4 + 8 + \dots + 2^{w-1} = 2^w$$

$$1 + \sum_{i=0}^{w-1} 2^i = 2^w$$

■ $w = 0$:

- $1 = 2^0$

■ Assume true for $w-1$:

- $1 + 1 + 2 + 4 + 8 + \dots + 2^{w-1} + 2^w = 2^w + 2^w = 2^{w+1}$

$$= 2^w$$

Negation: Complement & Increment

■ Claim: Following Holds for 2's Complement

$$\sim x + 1 == -x$$

■ Complement

- Observation: $\sim x + x == 1111\dots111 == -1$

$$\begin{array}{r}
 x \quad \boxed{1} \boxed{0} \boxed{0} \boxed{1} \boxed{1} \boxed{1} \boxed{0} \boxed{1} \\
 + \quad \sim x \quad \boxed{0} \boxed{1} \boxed{1} \boxed{0} \boxed{0} \boxed{0} \boxed{1} \boxed{0} \\
 \hline
 -1 \quad \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1} \boxed{1}
 \end{array}$$

■ Complete Proof?

Complement & Increment Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

x = 0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Arithmetic: Basic Rules

- **Unsigned ints, 2's complement ints are isomorphic rings: isomorphism = casting**

- **Left shift**
 - Unsigned/signed: multiplication by 2^k
 - Always logical shift

- **Right shift**
 - Unsigned: logical shift, div (division + round to zero) by 2^k
 - Signed: arithmetic shift
 - Positive numbers: div (division + round to zero) by 2^k
 - Negative numbers: div (division + round away from zero) by 2^k
Use biasing to fix

Properties of Unsigned Arithmetic

■ Unsigned Multiplication with Addition Forms Commutative Ring

- Addition is commutative group
- Closed under multiplication
$$0 \leq \text{UMult}_w(u, v) \leq 2^w - 1$$
- Multiplication Commutative
$$\text{UMult}_w(u, v) = \text{UMult}_w(v, u)$$
- Multiplication is Associative
$$\text{UMult}_w(t, \text{UMult}_w(u, v)) = \text{UMult}_w(\text{UMult}_w(t, u), v)$$
- 1 is multiplicative identity
$$\text{UMult}_w(u, 1) = u$$
- Multiplication distributes over addition
$$\text{UMult}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UMult}_w(t, u), \text{UMult}_w(t, v))$$

Properties of Two's Comp. Arithmetic

■ Isomorphic Algebras

- Unsigned multiplication and addition
 - Truncating to w bits
- Two's complement multiplication and addition
 - Truncating to w bits

■ Both Form Rings

- Isomorphic to ring of integers mod 2^w

■ Comparison to (Mathematical) Integer Arithmetic

- Both are rings
- Integers obey ordering properties, e.g.,

$$u > 0 \quad \Rightarrow \quad u + v > v$$

$$u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$$

- These properties are not obeyed by two's comp. arithmetic

$$TMax + 1 \quad == \quad TMin$$

$$15213 * 30426 \quad == \quad -10030$$

(16-bit words)