1. Here is a list of various kinds of integer knapsack problems. We start with the following given data.

   Given: A knapsack with integer capacity W.

   Given: n items with integer weights $0 < w_1 \leq w_2 \leq \ldots \leq w_n$.

   There are two kinds of knapsack problems below. One kind uses each item as many times as needed, and the other kind uses each item at most one time. The second kind is often called a 0-1 Knapsack Problem and it is a slightly more "difficult" problem than the first kind.

   (a) If we have as many of each item as we need, can we fill the knapsack to its exact capacity? (This is a boolean decision problem, so the result is either true or false.)

   (b) If we use each item at most once, can we fill the knapsack to its exact capacity? (This is also a boolean decision problem, so the result is either true or false.)

   (c) If we have as many of each item as we need, what is the maximum weight we can fit into the knapsack?

   (d) If we use each item at most once, what is the maximum weight we can fit into the knapsack?

   Now Suppose that we are given the following additional data.

   Given: n positive values, $v_1, v_2, \ldots, v_n$, one value for each item.

   (e) If we have as many of each item as we need, what is the maximum value we can fit into the knapsack if we must fill the knapsack to its exact capacity? (This problem will not have a solution if problem (a) above is false for this knapsack. However, if $w_1 = 1$, then this problem must have a solution.)

   (f) If we use each item at most once, what is the maximum value that we can fit into the knapsack if we must fill the knapsack to its exact capacity? (If problem (b) above is false for this knapsack, then this problem does not have a solution.)

   (g) If we have as many of each item as we need, what is the maximum value we can fit into the knapsack?

   (h) If we use each item at most once, what is the maximum value that we can fit into the knapsack?

   Note: Even when problems (e) and (f) have solutions, they need not be the same as the solutions to problems (g) and (h). For example, if $W = 7$, $n = 4$, and

$$\begin{array}{llll} w_1 = 2, & w_2 = 3, & w_3 = 4, & w_4 = 6 \\ v_1 = 2, & v_2 = 5, & v_3 = 3, & v_4 = 9.5 \end{array}$$

   then you can check that problems (e), (f), (g), and (h) all have different solutions.

   Note: If we assume that $w_i = i$, that is, the weights are $1, 2, 3, \ldots, n$, then problems (e) and (g) are the CutRod problem from the CLRS textbook, Section 15.1.

2. Here are recurrence relations for the above knapsack problems.

   (a) Let $K[i, j]$ be true if a knapsack of capacity j can be filled to capacity using each item from 1 to i as many times as needed (and false otherwise). Then

$$K[i, j] = K[i-1, j] \vee K[i, j-w_i]$$

   with initial conditions

$$\begin{aligned} K[i, 0] &= \text{true} \\ K[0, j] &= \text{false} \quad \text{if } j > 0 \\ K[i, j] &= \text{false} \quad \text{if } j < 0. \end{aligned}$$

   Notice that there are two ways that you can "fail" to solve this problem, either you overflow the knapsack (and so the capacity j becomes negative), or you have extra capacity but you have run out of items (so $j > 0$ but $i = 0$). And there is only one way that you can "succeed" in solving this problem, the capacity j becomes 0.

      i. If we don't like the idea of the capacity being allowed to become negative, we can write this recurrence relation and the initial conditions as follows:

$$K[i, j] = \begin{cases} K[i-1, j] \vee K[i, j-w_i] & \text{if } w_i \leq j \\ K[i-1, j] & \text{if } w_i > j \end{cases}$$

      with initial conditions

$$\begin{aligned} K[i, 0] &= \text{true} \\ K[0, j] &= \text{false} \quad \text{if } j > 0. \end{aligned}$$

      ii. We can solve this problem using a single-variable recurrence relation. Let $K[j]$ be true if a knapsack of capacity j can be filled to capacity using each item from 1 to n as many times as needed (and false otherwise). Then

$$K[j] = \bigvee_{i=1}^{n} K[j - w_i]$$

      with initial condition

$$\begin{aligned} K[0] &= \text{true} \\ K[j] &= \text{false} \quad \text{if } j < 0. \end{aligned}$$

   (b) Let $K[i, j]$ be true if a knapsack of capacity j can be filled to capacity using each item from 1 to i at most one time (and false otherwise). Then

$$K[i, j] = K[i-1, j] \vee K[i-1, j-w_i]$$

   with initial conditions

$$\begin{aligned} K[i, 0] &= \text{true} \\ K[0, j] &= \text{false} \quad \text{if } j < 0 \text{ or } j > 0. \end{aligned}$$

   This problem does not have a single-variable recurrence relation that solves it. This is what we mean when we say that the 0-1 Knapsack Problems are slightly harder. They require a much less obvious description of their appropriate sub-problems.

(c) Let $KW[i, j]$ be the optimal "knapsack weight" for a knapsack of capacity j using each item from 1 to i as many times as needed. Then

$$KW[i,j] = \begin{cases} \max\{\, KW[i-1,j], \quad w_i + KW[i, j-w_i] \,\} & \text{if } w_i \le j \\ \qquad KW[i-1,j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KW[i, 0] = 0$ and $KW[0, j] = 0$.

i. We can also write this recurrence relation and its initial conditions this way:

$$KW[i,j] = \max\{\, KW[i-1,j], \quad w_i + KW[i, j-w_i] \,\}$$

with initial conditions

$$KW[i,0] = 0 \quad \text{and} \quad KW[0,j] = \begin{cases} -\infty & \text{if } j < 0 \\ 0 & \text{if } j \ge 0 \end{cases}$$

ii. We can solve this problem using a single-variable recurrence relation. Let $KW[j]$ be the optimal "knapsack weight" for a knapsack of capacity j using each item from 1 to n as many times as needed. Then

$$KW[j] = \max_{w_i \le j}\{\, w_i + KW[j-w_i] \,\}$$

with initial condition $KW[0] = 0$. We are using the convention that the maximum over an empty set is 0 (we get an empty set when none of the items w_i fit in the knapsack with capacity j). Notice that the max in this recurrence relation would represent a loop, since this is a max over a set of indices i.

(d) Let $KW[i, j]$ be the optimal "knapsack weight" for a knapsack of capacity j using each item from 1 to i at most one time. Then

$$KW[i,j] = \begin{cases} \max\{\, KW[i-1,j], \quad w_i + KW[i-1, j-w_i] \,\} & \text{if } w_i \le j \\ \qquad KW[i-1,j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KW[i, 0] = 0$ and $KW[0, j] = 0$.

This problem does not have a single-variable recurrence relation that solves it.

(e) Let $KV[i, j]$ be the optimal "knapsack value" for a knapsack of capacity j that must be filled to its exact capacity using each item from 1 to i as many times as needed. Then

$$KV[i,j] = \begin{cases} \max\{\, KV[i-1, j], \quad v_i + KV[i, j - w_i] \,\} & \text{if } w_i \leq j \\ \qquad KV[i-1, j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KV[i, 0] = 0$ and $KV[0, j] = -\infty$.

A final value of $KV[n, W] = -\infty$ means that the problem does not have a solution. Notice that we are using the fact that $-\infty + x = -\infty$ for any number x.

i. We can solve this problem using a single-variable recurrence relation. Let $KV[j]$ be the optimal "knapsack value" for a knapsack of capacity j that must be filled to its exact capacity using each item from 1 to n as many times as needed. Then

$$KV[j] = \begin{cases} \max_{1 \leq i \leq n}\{\, v_i + KV[j - w_i] \,\} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \\ -\infty & \text{if } j < 0 \end{cases}$$

A final value $KV[W] = -\infty$ means that the problem does not have a solution.

(f) Let $KV[i, j]$ be the optimal "knapsack value" for a knapsack of capacity j that must be filled to its exact capacity using each item from 1 to i at most once. Then

$$KV[i,j] = \begin{cases} \max\{\, KV[i-1, j], \quad v_i + KV[i-1, j - w_i] \,\} & \text{if } w_i \leq j \\ \qquad KV[i-1, j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KV[i, 0] = 0$ and $KV[0, j] = -\infty$.

A final value of $KV[n, W] = -\infty$ means that the problem does not have a solution.

(g) Let $KV[i, j]$ be the optimal "knapsack value" for a knapsack of capacity j using each item from 1 to i as many times as needed. Then

$$KV[i,j] = \begin{cases} \max\{\, KV[i-1, j], \quad v_i + KV[i, j - w_i] \,\} & \text{if } w_i \leq j \\ \qquad KV[i-1, j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KV[i, 0] = 0$ and $KV[0, j] = 0$.

(h) Let $KV[i, j]$ be the optimal "knapsack value" for a knapsack of capacity j using each item from 1 to i at most one time. Then

$$KV[i,j] = \begin{cases} \max\{\, KV[i-1, j], \quad v_i + KV[i-1, j - w_i] \,\} & \text{if } w_i \leq j \\ \qquad KV[i-1, j] & \text{if } w_i > j \end{cases}$$

with initial conditions $KV[i, 0] = 0$ and $KV[0, j] = 0$.