1. An **indefinite integral**

$$\int f(x)\,dx$$

means "find all the functions whose derivative is f."

2. Given a function $f(x)$, **an antiderivative** of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

Notice that if $F(x)$ is an antiderivative of $f(x)$, then for any number c, the function $F(x) + c$ is also an antiderivative for $f(x)$ since $\frac{d}{dx}(F(x) + c) = f(x)$. So if $f(x)$ has one antiderivative, then $f(x)$ actually has an infinite number of antiderivatives.

3. The **indefinite integral of** $f(x)$ represents *all* possible antiderivatives of the function $f(x)$. So if $F(x)$ is one antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + c$$

4. Every differentiation rule can be reversed into an antidifferentiation rule. Here is a table showing several differentiation rules and the complementary antidifferentiation rules.

$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}a^x = a^x\ \ln(a)$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}x^n = n\,x^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$
$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$

5. **Linearity** of the indefinite integral.

$$\int [\, f(x) + g(x)\,]\, dx = \int f(x)\, dx \, + \int g(x)\, dx$$

$$\int k f(x)\, dx = k \int f(x)\, dx$$

Notice that these rules are true because of the linearity of the derivative.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx}(k\, f(x)) = k\, \frac{d}{dx} f(x)$$

6. The chain rule for differentiation

$$\frac{d}{dx} h(g(x)) = h'(g(x))\, g'(x)$$

gives rise to another antidifferentiation rule

$$\int h'(g(x))\, g'(x)\, dx = h(g(x))$$

which is usually written in a slightly different way by letting $f = h'$

$$\int f(g(x))\, g'(x)\, dx = F(g(x))$$

where $F(x)$ is any antiderivative for $f(x)$. This rule is what is behind the "Method of Substitution" for finding antiderivatives.

7. Here is one way to think about the **Method of Substitution**. If you are given an integral problem that has this pattern

$$\int f(g(x))\, g'(x)\, dx,$$

that is, the integral problem looks like there is an "outer function" f, an "inner function" g, and the derivative g' of the inner function, then you do the following.

Let u represent the inner function,

$$u = g(x)$$

so

$$\frac{du}{dx} = g'(x)$$

and so

$$du = g'(x)\, dx.$$

Now substitute into the original problem

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

which leaves you with *the much easier problem* $\int f(u)\, du$. (This is the goal of the substitution method, to turn a hard problem into an easier problem.)

Suppose the F is a solution to this easier problem (that is, F is an antiderivative for f, $F' = f$). So then you have

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u).$$

But the answer should not be a function of u, the answer should be in terms of the original variable x, so substitute back into the answer what u represents. Then you get

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) = F(g(x)).$$

Here is a summary of the u-substitution method.

$$\underbrace{\int f(g(x))\, g'(x)\, dx}_{\text{start with this problem}} \quad = \quad \underbrace{\int f(u)\, du}_{\text{make the } u\text{-substitution}} \quad = \quad \underbrace{F(u)}_{\text{solve the easier problem}} \quad = \quad \underbrace{F(g(x))}_{\text{undo the } u\text{-substitution}}$$

8. The product rule for differentiation

$$\frac{d}{dx}\left(f(x)\, g(x)\right) = f'(x)\, g(x) + f(x)\, g'(x)$$

gives rise to another antidifferentiation rule

$$\int f'(x)\, g(x) + f(x)\, g'(x)\, dx = f(x)\, g(x)$$

which we can rewrite in a slightly different way

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx.$$

If we let $u = f(x)$ and $dv = g'(x)dx$, the last equation can be written as

$$\int u\, dv = uv - \int v\, du$$

and this is called the "integration by parts" formula.