MA 26100 **Double Integrals and Iterated Integrals** Spring, 2016

1. Given a function of two variables $f(x,y)$ and a rectangle R in the xy-plane, the **double integral** of f over R represents the "signed volume" that is trapped between the graph of f and the rectangle R. Here is a way to "read" a double integral.

$$\underbrace{\iint_R \underbrace{\underbrace{f(x,y)}_{\text{height}}\,\underbrace{dA}_{\text{area}}}_{\text{small bit of volume}}}_{\text{total volume = sum of all the small volumes}}$$

   Start with $f(x,y)$ which represents the height of the function f at the point (x,y) (so its unit is the unit of length). The dA represents a small rectangular piece of area located at the point (x,y) (so dA has the units of *length*^{2}). Then $f(x,y)\,dA$ represents the small piece of volume over the small piece of area at the point (x,y), that is *height* $\times$ *area* $=$ *volume* (which has the unit of *length*^{3}). Finally, we "sum over" all of the little rectangular pieces of area that make up the rectangle R to get the "total volume," $\iint_R f(x,y)\,dA$. The integral sign, $\int$, is an elongated S and represents the verb "sum." The double integral, $\iint$, represents summing over *both* the rows and columns of small rectangles that make up the region R.

2. We evaluate (or compute) a double integral over a rectangle by converting it into an **iterated integral**. A double integral over a rectangle can be converted into two iterated integrals,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\;dy\;dx = \int_c^d \int_a^b f(x,y)\;dx\;dy$$

3. Here are two ways to "read" an iterated integral (you should draw a picture for each of these ways of reading the integral).

$$\underbrace{\int_a^b \underbrace{\underbrace{\left[\int_c^d \underbrace{\underbrace{f(x,y)}_{\text{height}}\,\underbrace{dy}_{\text{length}}}_{\text{area}}\right]}_{\text{total area of an } x\text{-slice}}\quad \underbrace{dx}_{\text{thickness}}}_{\text{area} \times \text{thickness} = \text{volume of an } x\text{-slice}}}_{\text{total volume = sum of all the } x\text{-slices}} \quad = \quad \underbrace{\int_a^b \underbrace{\int_c^d \underbrace{\left[\underbrace{f(x,y)}_{\text{height}}\, \underbrace{\underbrace{dy}_{\text{length}}\,\underbrace{dx}_{\text{width}}}_{\text{area}}\right]}_{\text{small piece of volume}}}_{\text{total volume of an } x\text{-slice}}}_{\text{total volume = sum of all the } x\text{-slices}}$$

   Remember that in the "inner integral" we are holding x fixed and are integrating with respect to y (that is what we mean by a x-slice). After you complete the inner integral, there should no longer be any y's in the integral, and you do the "outer integral" with respect to x.

   When you use a double integral formula to solve a physics or engineering problem, sometimes the way to read the integral on the left makes it easier to understand the problem, sometimes the way to read the integral on the right is better.