1. Given a function of two variables $f(x, y)$ and a "region" R that is a subset of the xy-plane, remember that the **double integral** of f over R represents the "signed volume" that is trapped between the graph of f and the region R. Here is a way to "read" the double integral.

$$\underbrace{\iint\limits_R \underbrace{\underbrace{f(x,y)}_{\text{height}}\,\underbrace{dA}_{\text{area}}}_{\text{small bit of volume}}}_{\text{total volume = sum of all the small volumes}}$$

   Start with $f(x, y)$ which represents the height of the function f at the point (x, y) (so its unit is the unit of length). The dA represents a small rectangular piece of area located at the point (x, y) (so dA has the units of *length*^{2}). Then $f(x, y)\,dA$ represents the small piece of volume over the small piece of area at the point (x, y), that is *height* $\times$ *area* = *volume* (which has the unit of *length*^{3}). Finally, we "sum over" all of the little rectangular pieces of area that make up the region R to get the "total volume," $\iint_R f(x, y)\,dA$. (The integral sign, $\int$, is an elongated S and represents the verb "sum." The double integral, $\iint$, represents summing over *both* the rows and columns of small rectangles that make up the region R.)

2. The double integral has many of the same properties as the single integral. The integral of a sum of two functions is the sum of two integrals.

$$\iint\limits_R f(x,y) + g(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA$$

   And the integral of a constant times a function is the constant times the integral (or, constants can factor out in front of the integral sign).

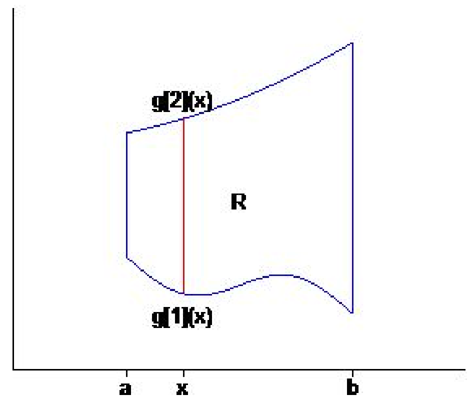
$$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$$

   If the region R is cut into two regions R_1 and R_2 (so we can say something like $R = R_1 + R_2$), then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA.$$

3. We evaluate (or compute) a double integral by converting it into an **iterated integral**. But in order to convert a double integral into an iterated integral, the region R must be a "nice" region in one of two senses.

   A region R is a **Type 1 region** if it looks something like this.



   That is, there are two numbers a and b and two functions $g_1(x)$ and $g_2(x)$ such that the region R can be described as all the points (x, y) with

$$\{\,(x, y) \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\,\}.$$

   If we are given a double integral over a Type 1 region, then we can evaluate the double integral by converting it into a Type 1 iterated integral.

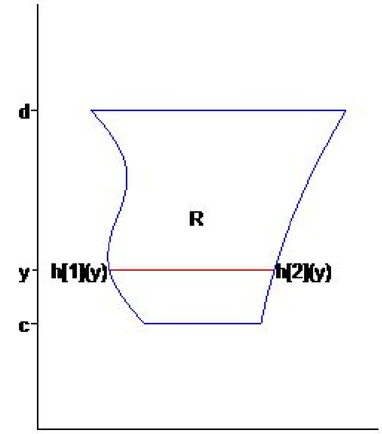
$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx$$

   Here are two ways to "read" this iterated integral.

$$\underbrace{\int_a^b \underbrace{\underbrace{\left[\int_{g_1(x)}^{g_2(x)} \underbrace{\underbrace{f(x,y)}_{\text{height}}\ \underbrace{dy}_{\text{length}}}_{\text{area}}\right]}_{\text{total area of an } x\text{-slice}} \quad \underbrace{dx}_{\text{thickness}}}_{\text{area} \times \text{thickness} = \text{volume of an } x\text{-slice}}}_{\text{total volume} = \text{sum of all the } x\text{-slices}} \quad = \quad \underbrace{\int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} \underbrace{\left[\underbrace{f(x,y)}_{\text{height}}\ \underbrace{\underbrace{dy}_{\text{length}}\ \underbrace{dx}_{\text{width}}}_{\text{area}}\right]}_{\text{small piece of volume}}}_{\text{total volume of an } x\text{-slice}}}_{\text{total volume} = \text{sum of all the } x\text{-slices}}$$

   Remember that in the "inner integral" we are holding x fixed and are integrating with respect to y (that is what we mean by a x-slice). After you complete the inner integral, there should no longer be any y's in the integral, and you do the "outer integral" with respect to x.

4. A region R is a **Type 2 region** if it looks something like this.



That is, there are two numbers c and d and two functions $h_1(y)$ and $h_2(y)$ such that the region R can be described as all the points (x, y) with

$$\{\, (x, y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \,\}.$$

If we are given a double integral over a Type 2 region, then we can evaluate the double integral by converting it into a Type 2 iterated integral.

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Here is a way to "read" this iterated integral.

$$\underbrace{\int_c^d \underbrace{\underbrace{\left[\int_{h_1(y)}^{h_2(y)} \underbrace{\underbrace{f(x,y)}_{\text{height}}\,\underbrace{dx}_{\text{length}}}_{\text{small bit of area}}\right]}_{\text{total area of a } y\text{-slice}}\;\underbrace{dy}_{\text{thickness}}}_{\text{area} \times \text{thickness} = \text{volume of a } y\text{-slice}}}_{\text{total volume} = \text{sum of all the } y\text{-slices}}$$

Remember that in the "inner integral" we are holding y fixed and are integrating with respect to x (that is what we mean by a y-slice). After you complete the inner integral, there should no longer be any x's in the integral, and you do the "outer integral" with respect to y.

5. Given a Type 1 or a Type 2 iterated integral, you should be able to reconstruct, from the limits of integration, what the region R looks like. (See problems 1–14 on page 691.)

6. Given a double integral over a region R, you should be able to determine if the region is a Type 1 or a Type 2 region (or possibly both) and convert the double integral into the appropriate iterated integral. (See problems 15–20 on page 691.)

7. Given a Type 1 iterated integral, you should be able to determine if the region R is also a Type 2 region and, if so, change the order of integration into a Type 2 iterated integral.

   Given a Type 2 iterated integral, you should be able to determine if the region R is also a Type 1 region and, if so, change the order of integration into a Type 1 iterated integral. (See problems 33–38 on page 692.)

8. Given a point (x, y) in the xy-plane, we can find that point's **polar coordinates** by using the conversion formulas

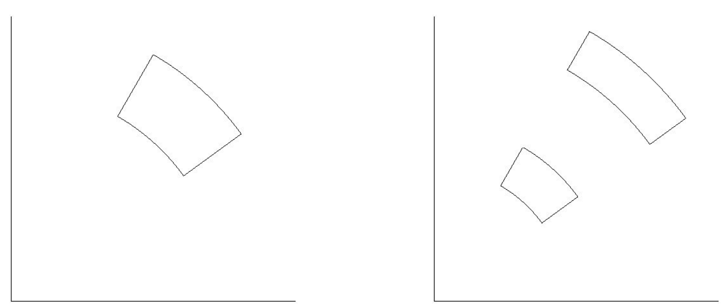
$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x).$$

   (Remember that the second formula is only correct in the first quadrant of the plane.)

   Given the (r, θ) coordinates of a point in the plane, we can find that point's rectangular coordinates by using the conversion formulas

$$x = r\,\cos(\theta) \qquad y = r\,\sin(\theta).$$

9. Before we can do double integrals in polar coordinates, we need to know what the element of area, dA, is in polar coordinates. A "polar rectangle," with dimensions dr and $d\theta$, looks like the following picture on the left.

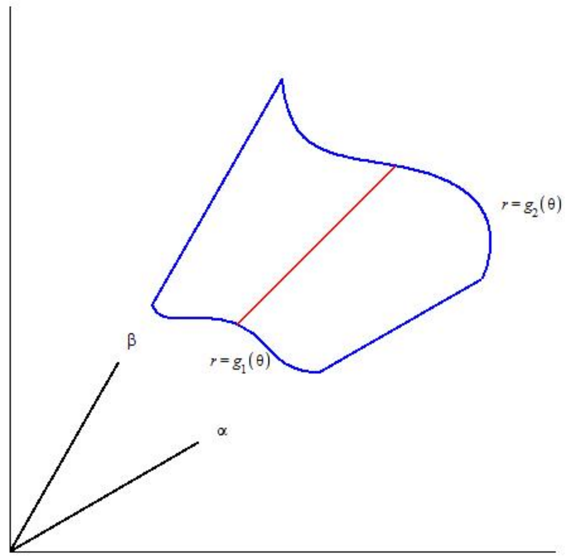


   The area of this "polar rectangle" is *not* $dr d\theta$. The picture on the right shows why. This picture shows two polar rectangles with the exact same dimensions, the same dr and the same $d\theta$. But one polar rectangle has larger area than the other because it is further from the origin. The area of a polar rectangle with dimensions dr and $d\theta$ is directly proportional to its distance r from the origin. So the area, dA, of the polar element of area is

$$dA = r\,dr\,d\theta.$$

   For the polar rectangles in the pictures, the value of r is the average of the inner and outer radiuses of each rectangle.

10. A region R is a **polar region** (the textbook calls this a r-simple region) if it looks something like this.



That is, there are two angles α and β and two functions $g_1(\theta)$ and $g_2(\theta)$ such that the region R can be described as all the points with polar coordinates (r, θ) with

$$\{\,(r,\theta) \mid \alpha \le \theta \le \beta \text{ and } g_1(\theta) \le r \le g_2(\theta)\,\}.$$

If we are given a double integral over a polar region, then we can evaluate the double integral by converting it into a polar iterated integral.

$$\iint_R f(x,y)\,dA = \underbrace{\int_\alpha^\beta \underbrace{\int_{g_1(\theta)}^{g_2(\theta)} \underbrace{\underbrace{f(r\cos\theta, r\sin\theta)}_{\text{height}}\ \underbrace{r\,dr\,d\theta}_{\text{polar area}}}_{\text{small bit of volume}}}_{\text{total volume of a }\theta\text{-slice}}}_{\text{total volume}}$$

In the inner integral we are holding θ fixed and are integrating with respect to r. After you complete the inner integral, there should no longer be any r's in the integral, and you do the outer integral with respect to θ.

Be sure to remember to convert the function f from a function of rectangular coordinates, $f(x, y)$, into a function of polar coordinates, $f(r\cos\theta, r\sin\theta)$. This means that every occurrence of x in the original formula for f should be replaced with $r\cos\theta$ and every occurrence of y should be replaced with $r\sin\theta$. (Sometimes the function f is given to you already in polar coordinates, $f(r, \theta)$, and doesn't need to be converted.)

11. Given a region R in the xy-plane and a density function, $\delta(x, y)$, for the region, we can compute the mass of the region and the region's center of mass.

    The **mass** of a region R with density function $\delta(x, y)$ is given by

$$m = \iint_R \delta(x, y)\, dA.$$

    Recall that the units for density are (unit of mass)/(unit of area), and the value of $\delta(x, y)$ should be read as "$\delta(x, y)$ units of mass *per* unit of area." Here is the way to "read" the integral for mass.

$$m = \underbrace{\iint_R \qquad \underbrace{\underbrace{\delta(x, y)}_{\frac{mass}{area}} \underbrace{dA}_{area}}_{\text{mass of one small element}}}_{\text{total mass}}$$

    The **center of mass**, for a region R with density function $\delta(x, y)$, is a point $(\bar{x}, \bar{y})$ where the region would be "balanced." The formulas for the two coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\,\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}$$

    and

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\,\delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}.$$

12. The area of the graph of a function $f(x, y)$ over a region R is given by the double integral

$$\iint_R \sqrt{f_x^2(x, y) + f_y^2(x, y) + 1}\; dA$$

    Recall that the integrand, $\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}$, is the length of the vector normal to the graph at the point $(x, y, f(x, y))$. The normal vector is $f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} + \mathbf{k}$, and we get that normal vector by computing the cross product of the two tangent vectors $(1,\ 0,\ f_x(x, y))$ and $(0,\ 1,\ f_y(x, y))$. The length of a cross product is the area of the parallelogram formed by the two vectors. In this case, the parallelogram is formed by the two tangent vectors (scaled by dx and dy respectively) and the area of the parallelogram is an approximation for the area of one small patch of the graph of f (the patch that is right above the point (x, y)). The double integral sums up all of the small parallelogram approximations to get the total area of the surface.

13. Given a function of three variables $f(x, y, z)$, and a volume V that is a subset of the three-dimensional xyz-space, the **triple integral** of f over V is written as

$$\iiint_V f(x, y, z)\, dV.$$

14. If the volume V in a triple integral is a rectangular box, with $a \le x \le b$ and $c \le y \le d$ and $p \le z \le q$, then the triple integral of $f(x, y, z)$ over the volume V can be evaluated as a triple iterated integral,

$$\iiint_V f(x, y, z)\, dV = \int_a^b \int_c^d \int_p^q f(x, y, z)\, dz\, dy\, dx.$$

When the volume V is a rectangular box (but *only* when it is a rectangular box), we can in fact use any one of six different orderings of the iterated integrals. For example, we could also use

$$\iiint_V f(x, y, z)\, dV = \int_c^d \int_p^q \int_a^b f(x, y, z)\, dx\, dz\, dy.$$

15. A volume V is a **Type 1 volume** if there is a region R in the xy-plane and two functions $\phi_1(x, y)$ and $\phi_2(x, y)$ such that the volume V can be described as all the points (x, y, z) with

$$\{(x, y, z) \mid (x, y) \text{ is in } R \text{ and } \phi_1(x, y) \le z \le \phi_2(x, y)\}.$$

(See Figure 3 on page 709 or Figure 6 on page 711.) This means that the three-dimensional volume has the region R as its "shadow" in the xy-plane, and it has the graph of $\phi_1(x, y)$ as the bottom of the volume, and the graph of $\phi_2(x, y)$ as the top of the volume.

16. If the volume V in a triple integral is a Type 1 volume, then the triple integral can be evaluated as the following iterated integral.

$$\iiint_V f(x, y, z)\, dV = \iint_R \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, dz \right] dA.$$

If the region R is a Type 1 region, then the triple integral can be evaluated as the following triple iterated integral.

$$\iiint_V f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

17. In **polar coordinates**,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad dA = r\,dr\,d\theta.$$

18. In **cylindrical coordinates**,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad \text{and} \quad dV = r\,dz\,dr\,d\theta.$$

19. In **spherical coordinates**,

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi, \quad \text{and} \quad dV = \rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi.$$

20. Given a triple integral of a function $f(x, y, z)$ over a volume V, we can convert the triple integral from rectangular coordinates to cylindrical coordinates,

$$\iiint\limits_V f(x, y, z)\,dV = \iiint\limits_V f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

or we can convert the triple integral from rectangular to spherical coordinates,

$$\iiint\limits_V f(x, y, z)\,dV = \iiint\limits_V f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi.$$

Notice that each of the integrals on the right hand sides is still a triple integral (in cylindrical or spherical coordinates), not yet an iterated integral. To make each triple integral into an iterated integral, you need to choose an order for the iterated integrals and then determine limits of integration, using the chosen coordinate system, that, in the chosen order, describe the given volume V.

21. Spherical coordinates are often used to describe triple integrals over parts of a sphere. You should know how to write the limits of integration in spherical coordinates for a triple integral over the following parts of a sphere centered at the origin.

   - Any one of the left, right, front, back, top, or bottom halves of the sphere.
   - The part of the sphere contained in any one of the eight octants of xyz-space.
   - The volume between two spheres (a "spherical shell").
   - An "ice cream cone" shape (see Figure 9 on page 718 of the textbook).
   - An orange (or apple) slice (more formally, a **spherical wedge**).