1. Given a function of two variables $f(x, y)$ and a "region" $R$ that is a subset of the $xy$-plane, remember that the **double integral** of $f$ over $R$ represents the "signed volume" that is trapped between the graph of $f$ and the region $R$. Here is a way to "read" the double integral.

$$\underbrace{\iint\limits_R \underbrace{\underbrace{f(x,y)}_{\text{height}}\,\underbrace{dA}_{\text{area}}}_{\text{small bit of volume}}}_{\text{total volume = sum of all the small volumes}}$$

Start with $f(x, y)$ which represents the height of the function $f$ at the point $(x, y)$ (so its unit is the unit of length). The $dA$ represents a small rectangular piece of area located at the point $(x, y)$ (so $dA$ has the units of $length^2$). Then $f(x, y)\,dA$ represents the small piece of volume over the small piece of area at the point $(x, y)$, that is $height \times area = volume$ (which has the unit of $length^3$). Finally, we "sum over" all of the little rectangular pieces of area that make up the region $R$ to get the "total volume," $\iint_R f(x, y)\,dA$. (The integral sign, $\int$, is an elongated S and represents the verb "sum." The double integral, $\iint$, represents summing over *both* the rows and columns of small rectangles that make up the region $R$.)

2. The double integral has many of the same properties as the single integral. The integral of a sum of two functions is the sum of two integrals.

$$\iint\limits_R f(x,y) + g(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA$$

And the integral of a constant times a function is the constant times the integral (or, constants can factor out in front of the integral sign).

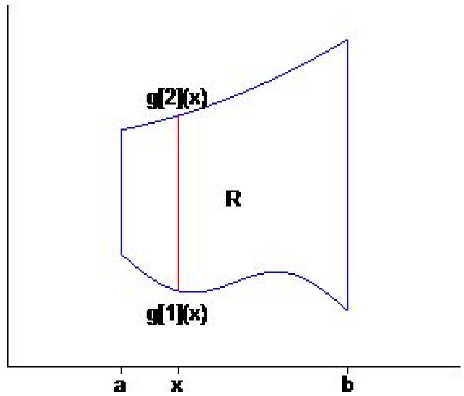
$$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$$

If the region $R$ is cut into two regions $R_1$ and $R_2$ (so we can say something like $R = R_1 + R_2$), then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA.$$

3. We evaluate (or compute) a double integral by converting it into an **iterated integral**. But in order to convert a double integral into an iterated integral, the region $R$ must be a "nice" region in one of two senses.

   A region $R$ is a **Type 1 region** if it looks something like this.




   That is, there are two numbers $a$ and $b$ and two functions $g_1(x)$ and $g_2(x)$ such that the region $R$ can be described as all the points $(x, y)$ with

$$\{\, (x, y) \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(x) \,\}.$$

   If we are given a double integral over a Type 1 region, then we can evaluate the double integral by converting it into a Type 1 iterated integral.

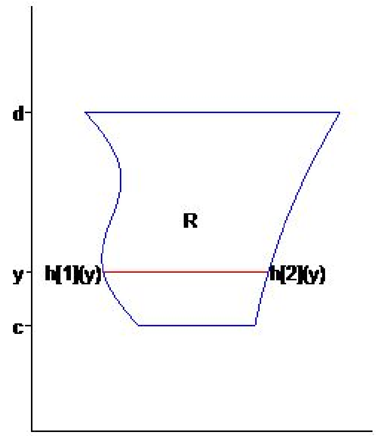
$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx$$

   Here are two ways to "read" this iterated integral.

$$\underbrace{\int_a^b \underbrace{\underbrace{\left[\int_{g_1(x)}^{g_2(x)} \underbrace{\underbrace{f(x,y)}_{\text{height}}\ \underbrace{dy}_{\text{length}}}_{\text{area}}\right]}_{\text{total area of an } x\text{-slice}} \quad \underbrace{dx}_{\text{thickness}}}_{\text{area} \times \text{thickness} = \text{volume of an } x\text{-slice}}}_{\text{total volume} = \text{sum of all the } x\text{-slices}} \quad = \quad \underbrace{\int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} \underbrace{\left[\underbrace{f(x,y)}_{\text{height}}\ \underbrace{\underbrace{dy}_{\text{length}}\ \underbrace{dx}_{\text{width}}}_{\text{area}}\right]}_{\text{small piece of volume}}}_{\text{total volume of an } x\text{-slice}}}_{\text{total volume} = \text{sum of all the } x\text{-slices}}$$

   Remember that in the "inner integral" we are holding $x$ fixed and are integrating with respect to $y$ (that is what we mean by a $x$-slice). After you complete the inner integral, there should no longer be any $y$'s in the integral, and you do the "outer integral" with respect to $x$.

4. A region $R$ is a **Type 2 region** if it looks something like this.




That is, there are two numbers $c$ and $d$ and two functions $h_1(y)$ and $h_2(y)$ such that the region $R$ can be described as all the points $(x, y)$ with

$$\{\, (x, y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \,\}.$$

If we are given a double integral over a Type 2 region, then we can evaluate the double integral by converting it into a Type 2 iterated integral.

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Here is a way to "read" this iterated integral.

$$\underbrace{\int_c^d \underbrace{\underbrace{\left[ \int_{h_1(y)}^{h_2(y)} \underbrace{\underbrace{f(x,y)}_{\text{height}}\ \underbrace{dx}_{\text{length}}}_{\text{small bit of area}} \right]}_{\text{total area of a } y\text{-slice}} \quad \underbrace{dy}_{\text{thickness}}}_{\text{area} \times \text{thickness} = \text{volume of a } y\text{-slice}}}_{\text{total volume} = \text{sum of all the } y\text{-slices}}$$

Remember that in the "inner integral" we are holding $y$ fixed and are integrating with respect to $x$ (that is what we mean by a $y$-slice). After you complete the inner integral, there should no longer be any $x$'s in the integral, and you do the "outer integral" with respect to $y$.

5. Given a Type 1 or a Type 2 iterated integral, you should be able to reconstruct, from the limits of integration, what the region $R$ looks like. (See problems 1–14 on page 691.)

6. Given a double integral over a region $R$, you should be able to determine if the region is a Type 1 or a Type 2 region (or possibly both) and convert the double integral into the appropriate iterated integral. (See problems 15–20 on page 691.)

7. Given a Type 1 iterated integral, you should be able to determine if the region $R$ is also a Type 2 region and, if so, change the order of integration into a Type 2 iterated integral.

   Given a Type 2 iterated integral, you should be able to determine if the region $R$ is also a Type 1 region and, if so, change the order of integration into a Type 1 iterated integral. (See problems 33–38 on page 692.)

8. Given a point $(x, y)$ in the $xy$-plane, we can find that point's **polar coordinates** by using the conversion formulas

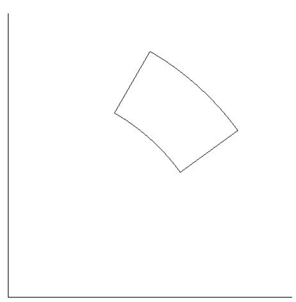
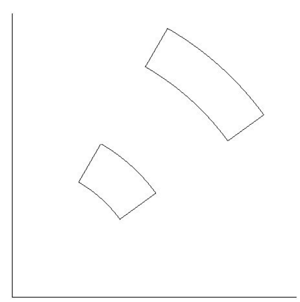
   $$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x).$$

   (Remember that the second formula is only correct in the first quadrant of the plane.)

   Given the $(r, \theta)$ coordinates of a point in the plane, we can find that point's rectangular coordinates by using the conversion formulas

   $$x = r\,\cos(\theta) \qquad y = r\,\sin(\theta).$$

9. Before we can do double integrals in polar coordinates, we need to know what the element of area, $dA$, is in polar coordinates. A "polar rectangle," with dimensions $dr$ and $d\theta$, looks like the following picture on the left.

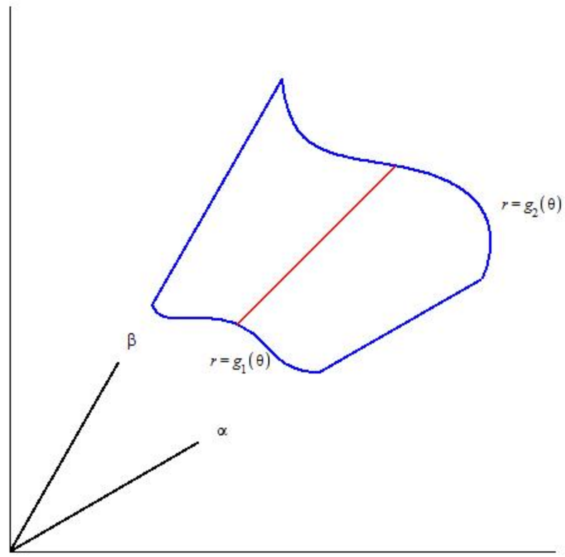
   
   

   The area of this "polar rectangle" is *not* $dr d\theta$. The picture on the right shows why. This picture shows two polar rectangles with the exact same dimensions, the same $dr$ and the same $d\theta$. But one polar rectangle has larger area than the other because it is further from the origin. The area of a polar rectangle with dimensions $dr$ and $d\theta$ is directly proportional to its distance $r$ from the origin. So the area, $dA$, of the polar element of area is

   $$dA = r\,dr\,d\theta.$$

   For the polar rectangles in the pictures, the value of $r$ is the average of the inner and outer radiuses of each rectangle.

10. A region $R$ is a **polar region** (the textbook calls this a $r$-simple region) if it looks something like this.




That is, there are two angles $\alpha$ and $\beta$ and two functions $g_1(\theta)$ and $g_2(\theta)$ such that the region $R$ can be described as all the points with polar coordinates $(r, \theta)$ with

$$\{\,(r,\theta) \mid \alpha \le \theta \le \beta \text{ and } g_1(\theta) \le r \le g_2(\theta)\,\}.$$

If we are given a double integral over a polar region, then we can evaluate the double integral by converting it into a polar iterated integral.

$$\iint\limits_R f(x,y)\,dA = \underbrace{\int_\alpha^\beta \underbrace{\int_{g_1(\theta)}^{g_2(\theta)} \underbrace{\underbrace{f(r\cos\theta, r\sin\theta)}_{\text{height}}\ \underbrace{r\,dr\,d\theta}_{\text{polar area}}}_{\text{small bit of volume}}}_{\text{total volume of a }\theta\text{-slice}}}_{\text{total volume}}$$

In the inner integral we are holding $\theta$ fixed and are integrating with respect to $r$. After you complete the inner integral, there should no longer be any $r$'s in the integral, and you do the outer integral with respect to $\theta$.

Be sure to remember to convert the function $f$ from a function of rectangular coordinates, $f(x, y)$, into a function of polar coordinates, $f(r\cos\theta, r\sin\theta)$. This means that every occurrence of $x$ in the original formula for $f$ should be replaced with $r\cos\theta$ and every occurrence of $y$ should be replaced with $r\sin\theta$. (Sometimes the function $f$ is given to you already in polar coordinates, $f(r, \theta)$, and doesn't need to be converted.)

11. Given a region $R$ in the $xy$-plane and a density function, $\delta(x, y)$, for the region, we can compute the mass of the region and the region's center of mass.

    The **mass** of a region $R$ with density function $\delta(x, y)$ is given by

$$m = \iint_R \delta(x, y)\, dA.$$

    Recall that the units for density are (unit of mass)/(unit of area), and the value of $\delta(x, y)$ should be read as "$\delta(x, y)$ units of mass *per* unit of area." Here is the way to "read" the integral for mass.

$$m = \underbrace{\iint_R \quad \underbrace{\underbrace{\delta(x, y)}_{\frac{mass}{area}} \underbrace{dA}_{area}}_{\text{mass of one small element}}}_{\text{total mass}}$$

    The **center of mass**, for a region $R$ with density function $\delta(x, y)$, is a point $(\bar{x}, \bar{y})$ where the region would be "balanced." The formulas for the two coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\, \delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}$$

    and

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\, \delta(x, y)\, dA}{\iint_R \delta(x, y)\, dA}.$$